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AMALGAMATION OF POLYADIC ALGEBRAS(1)

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Abstract. The main result of the paper is that for I an infinite set, the class of polyadic I-algebras (with equality) has the strong amalgamation property; i.e., if two polyadic I-algebras have a given common subalgebra they can be embedded in another algebra in such a way that the intersection of the images of the two algebras is the given common subalgebra.

Polyadic algebras were introduced by Halmos to provide an algebraic reflection of the study of first order logic without equality; later the algebras were enriched to allow the discussion of equality. That the notion is an adequate reflection of first order logic was demonstrated by Halmos' representation theorem for locally finite polyadic algebras of infinite degree (with or without equality). Daigneault and Monk have proved a strong extension of Halmos' theorem; namely, every polyadic algebra of infinite degree (without equality) is representable. Thus the notion of polyadic algebra is an adequate reflection of Keisler's predicate logic having infinitary predictates.

It is an interesting question to ask for algebraic versions of various model theoretic results. Daigneault has been successful in stating and proving algebraic versions of Beth's and Craig's theorems. This was done by proving the algebraic analogue of Robinson's consistency theorem: *Locally finite* polyadic I-algebras (with equality) of infinite degree have the amalgamation property. The major result of the present work is to remove the locally finite condition from Daigneault's result. With the stronger result, Robinson's, Beth's, and Craig's theorems follow for Keisler's logic though we shall defer this to a later paper.

We shall preface our work with an outline of the basic theory of polyadic algebras including theorems of Halmos [10] and important dilation and compression results of Daigneault and Monk [5].

Our set theoretic notation is standard, but it is perhaps worthwhile to outline some of our conventions. If X and Y are two sets, we write ${}^{Y}X$ for the set of all functions from Y into X. We shall often identify ${}^{2}X$ with $X \times X$—the cartesian

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product of X with itself. The power set of a set X is denoted by $S(X)$; thus $S(X) = \{Y : Y \subseteq X\}$. For f a function, dmn f and rng f denote the domain and range of f. We assume that ordinals have been defined so that an ordinal is the set of smaller ordinals. If α is an ordinal, the successor of α is denoted by $\alpha+1$. ω denotes the set of natural numbers; thus ω is the first infinite ordinal. Cardinals are initial ordinals. If α is a cardinal, the cardinal successor of α is denoted by α^+. For X a set, $|X|$ is the power or cardinal number of X. For α a cardinal, we define $2^\alpha = |{}^\alpha 2|$. If I is a set and X_i is a set for each $i \in I$, then $\mathsf{P}_{i\in I} X_i$ denotes the direct product of $\{X_i : i \in I\}$.

1. **Polyadic algebras.** In this section the basic definitions and theorems of the theory of polyadic algebras is given. For more details, the reader may consult Halmos [8] and [9] and Daigneault-Monk [5]. We shall use the notion of polyadic set algebra in place of Halmos' 0-valued functional algebra; the correspondence between these notions is easily established.

A *quantifier* on a Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ is a one-place operation $\exists$ satisfying for all $x, y \in A$:

(Q1) $\exists 0 = 0$;

(Q2) $x \leqq \exists x$;

(Q3) $\exists (x \cdot \exists y) = \exists x \cdot \exists y$.

THEOREM 1.1. *If $\exists$ is a quantifier on a Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$, then for all $x, y \in A$,*

(i) $\exists 1 = 1$;

(ii) $\exists\exists x = \exists x$;

(iii) *if $x \leqq y$, then $\exists x \leqq \exists y$;*

(iv) $\exists(-\exists x) = -\exists x$;

(v) $\exists(x+y) = \exists x + \exists y$.

DEFINITION 1.2. *For any set I, a polyadic I-algebra (PA_I) is an algebra of type $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S(\tau), \exists(J)\rangle_{\tau\in {}^I I, J\subseteq I}$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, $S(\tau)$ is a Boolean endomorphism, $\exists(J)$ is a quantifier, and for all $x \in A$, $\sigma, \tau \in {}^I I$, and $J \subseteq I$, the following six conditions hold*:

(P1) $S(\delta_I)x = x$;

(P2) $S(\sigma\tau)x = S(\sigma)S(\tau)x$;

(P3) $\exists(0)x = x$;

(P4) $\exists(J \cup K)x = \exists(J)\exists(K)x$;

(P5) *if $\sigma \restriction I \sim J = \tau \restriction I \sim J$, then $S(\sigma)\exists(J)x = S(\tau)\exists(J)x$;*

(P6) *if $\sigma \restriction \sigma^{-1}J$ is one-one, then $\exists(J)S(\sigma)x = S(\sigma)\exists(\sigma^{-1}J)x$.*

$|I|$ is called the *degree* of the algebra. When no confusion is likely, we shall write $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S(\tau), \exists(J)\rangle$; thus it is assumed the $S(\tau)$ run over all $\tau \in {}^I I$ and similarly for $\exists(J)$.

Concrete examples of $\boldsymbol{PA}_I$'s may be obtained as follows. For U a nonempty set, $X \subseteq {}^I U$, $J \subseteq I$, and $\tau \in {}^I I$ we define

$$\exists (J)X = \{f \in {}^I U : \text{for some } g \in X,\ g \restriction I \sim J = f \restriction I \sim J\}$$

and

$$S(\tau)X = \{f \in {}^I U : f\tau \in X\}.$$

Suppose A is a collection of subsets closed under $\cup$, $\sim$, $S(\tau)$, $\exists (J)$ for all $\tau \in {}^I I$ and $J \subseteq I$; then $\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, {}^I U, S(\tau), \exists (J)\rangle$ is called a *polyadic set I-algebra* ($\boldsymbol{PSA}_I$). U is called the *domain* or *base* of A.

THEOREM 1.3. *A* $\boldsymbol{PSA}_I$ *is a* $\boldsymbol{PA}_I$.

For $\mathfrak{A}$ a $\boldsymbol{PA}_I$, $\mathfrak{A}$ is representable (an $\boldsymbol{RPA}_I$) if it is isomorphic to a subdirect product of $\boldsymbol{PSA}_I$'s. It is known [5] that for I infinite, every $\boldsymbol{PA}_I$ is representable. In §4, we give a slightly novel proof of this theorem.

DEFINITION 1.4. *Let* $\mathfrak{A}$ *be a* $\boldsymbol{PA}_I$, $x \in A$, *and* $J \subseteq I$; *then* x *is* ***independent*** *of* J *if* $\exists (J)x = x$. *Dually,* J ***supports*** x *if* x *is independent of* $I \sim J$.

The next theorem gives important properties of these concepts.

THEOREM 1.5. *Let* $\mathfrak{A}$ *be a* $\boldsymbol{PA}_I$; *then*

(i) *For* $x \in A$, $\{J : J \text{ supports } x\}$ *is a filter and* $\{J : x \text{ is independent of } J\}$ *is an ideal in* $S(I)$;

(ii) *For* $J \subseteq I$, $\{x : J \text{ supports } x\}$ *and* $\{x : x \text{ is independent of } J\}$ *are Boolean subalgebras of* A;

(iii) *For* $x \in A$, $K \subseteq I$, *if* x *is independent of* J, *then* $\exists (K)x = \exists (K \sim J)x$;

(iv) *For* $x \in A$, $K \subseteq I$, *if* J *supports* x, *then* $\exists (K)x = \exists (J \cap K)x$;

(v) *For* $x \in A$, $K \subseteq I$, $\exists (K)x$ *is independent of* K;

(vi) *For* $x \in A$, $K \subseteq I$, *if* J *supports* x, *then* $J \sim K$ *supports* $\exists (K)x$;

(vii) *For* $x \in A$, $\sigma, \tau \in {}^I I$, *if* J *supports* x *and* $\sigma \restriction J = \tau \restriction J$, *then* $S(\sigma)x = S(\tau)x$;

(viii) *For* $x \in A$, $\sigma \in {}^I I$, *if* J *supports* x, *then* $\sigma(J)$ *supports* $S(\sigma)x$.

We define universal quantifiers $\forall(J)$ on a $\boldsymbol{PA}_I$ $\mathfrak{A}$ for each $J \subseteq I$ by $\forall(J)x = -\exists (J) - x$. This notion is dual to that of existential quantifier.

DEFINITION 1.6. *For* I *any set, a polyadic equality I-algebra (a* $\boldsymbol{PEA}_I$*) is an algebra of type*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S(\tau), \exists (J), d_{ij}\rangle_{\tau \in {}^I I, J \subseteq I, i, j \in I}$$

such that $\langle A, +, \cdot, -, 0, 1, S(\tau), \exists (J)\rangle$ *is a* $\boldsymbol{PA}_I$, *for each* i, j, $d_{ij} \in A$, *and for* $i, j \in I$, $x \in A$, $\tau \in {}^I I$:

(i) $d_{ii} = 1$;

(ii) $S(\tau)d_{ij} = d_{i\tau j}$;

(iii) $x \cdot d_{ij} \leqq S(i/j)x$.

Here and throughout, (i/j) is the function from I into I which sends j to i and leaves the rest of the elements of I fixed. Similarly, (i, j) is the function which

interchanges i and j and leaves the rest of the elements fixed. The ambiguity which may arise when we consider more than one I should cause no difficulty. As for $\boldsymbol{PA}_I$'s, we shall write $\mathfrak{A}=\langle A, +, \cdot, -, 0, 1, S(\tau), \exists(J), d_{ij}\rangle$ when confusion is unlikely.

THEOREM 1.7. *Let $\mathfrak{A}$ be a $\boldsymbol{PEA}_I$, $x \in A$, $i, j, k \in I$, then*

(i) $S(i/j)d_{ij}=\exists(\{i\})d_{ij}=1$;

(ii) $d_{ij}=d_{ji}$;

(iii) $\{i, j\}$ *supports* d_{ij};

(iv) *if* $k \neq i, j$, *then* $\exists(\{k\})(d_{ik}\cdot d_{kj})=d_{ij}$;

(v) $x\cdot d_{ij}=S(i/j)x\cdot d_{ij}=S(i,j)x\cdot d_{ij}$;

(vi) $\exists(\{j\})(x\cdot d_{ij})=S(i/j)x$;

(vii) *if* $J\subseteq I$ *is finite*, $\exists(K)\prod_{i,j\in J}d_{ij}=1$ *whenever* $|J\sim K|\leqq 1$.

If U is a nonempty set and I any set, then for $i, j \in I$ we let $D_{ij}^{UI}=\{f\in {}^IU : f_i=f_j\}$. We will usually write D_{ij} for D_{ij}^{UI}. *A polyadic equality set I-algebra* (a $\boldsymbol{PESA}_I$) is a $\boldsymbol{PSA}_I$ which contains D_{ij} for all $i, j \in I$.

THEOREM 1.8. *A $\boldsymbol{PESA}_I$ is a $\boldsymbol{PEA}_I$.*

A *representable* $\boldsymbol{PEA}_I$ (a $\boldsymbol{RPEA}_I$) is a $\boldsymbol{PEA}_I$ isomorphic to a subdirect product of $\boldsymbol{PESA}_I$'s. Unlike $\boldsymbol{PA}_I$'s, there are nonrepresentable $\boldsymbol{PEA}_I$'s for every I having at least two elements. (There are nonrepresentable $\boldsymbol{PA}_I$'s only for $3\leqq|I|<\omega$.)

Of course all of the properties of $\boldsymbol{PA}_I$'s mentioned so far (except for representability) carry over to $\boldsymbol{PEA}_I$'s. This also applies to notions and results mentioned throughout the remainder of this section.

For $\mathfrak{A}$ a $\boldsymbol{PA}_I$, an *ideal* of $\mathfrak{A}$ is a Boolean ideal M of $\mathfrak{A}$ such that for every $x \in M$, $\exists(I)x \in M$. It is easily seen that an ideal of A is closed under $\exists(J)$ and $S(\tau)$ for all $J\subseteq M$ and $\tau \in {}^II$. The ideals of a $\boldsymbol{PA}_I$ have exactly the same relationship to its congruence relations as the ideals of a Boolean algebra have to its congruence relations. If M is an ideal of a $\boldsymbol{PA}_I$ $\mathfrak{A}$, we will write $\mathfrak{A}/M$ for the algebra obtained by factoring out M in a manner exactly analogous to this operation for Boolean algebras. It is then easy to show that just as for Boolean algebras, every $\boldsymbol{PA}_I$ is semisimple, i.e. isomorphic to a subdirect product of simple $\boldsymbol{PA}_I$'s.

An important concept is that of local degree. For $\mathfrak{A}$ a $\boldsymbol{PA}_I$, the *local degree* of $\mathfrak{A}$ is the smallest infinite cardinal $\mathfrak{M}$ such that for every $x \in A$, x has a support J with $|J|<\mathfrak{M}$. If the local degree of $\mathfrak{A}$ is ω, $\mathfrak{A}$ is said to be *locally finite*. Clearly if I is infinite, $\mathfrak{M}\leqq|I|^+$. It is easy to see that every possible local degree is realized; namely

THEOREM 1.9. *Let I be infinite and $\mathfrak{M}$ be an infinite cardinal less than or equal to $|I|^+$. Furthermore let U be a set with at least two elements. Then if A is the family of all subsets of ${}^IU$ with a support of power less than $\mathfrak{M}$,*

$$\mathfrak{A}=\langle A, \cup, \cap, \sim, 0, {}^IU, S(\tau), \exists(J)\rangle_{\tau\in{}^II, J\subseteq I}$$

is a $\boldsymbol{AP}_I$ of local degree $\mathfrak{M}$.

The next theorem will be useful in the proof of our main result.

THEOREM 1.10. *Let I be any set, $\mathfrak{M}$ an infinite cardinal, U a nonempty set. Suppose $\mathfrak{A}$ is the $\boldsymbol{PA}_I$ of subsets of ${}^IU$ with support of power less that $\mathfrak{M}$. For γ a permutation of U and $X \in A$, let $\hat{\gamma}X$ be the set of γf such that $f \in X$. Then the mapping $\gamma \mapsto \hat{\gamma}$ is an isomorphism of the group of all permutations of U onto the group of automorphisms of $\mathfrak{A}$.*

Proof. For γ a permutation of U, it is clear that $\hat{\gamma}$ is an automorphism of $\mathfrak{A}$. Obviously $(\gamma\gamma')=\hat{\gamma}\hat{\gamma}'$ if γ and γ' are permutations of U. Now if γ is not the identity on U, say $\gamma u \neq u$, then taking $X=\{f \in {}^IU : f_i=u\}$ for some $i \in I$ with $\gamma_i \neq i$, we have $\hat{\gamma}X \neq X$. Thus the mapping $\gamma \mapsto \hat{\gamma}$ is one to one.

Now suppose Φ is an automorphism of $\mathfrak{A}$. For $i \in I$, let $\mathfrak{A}^i$ be the Boolean subalgebra of $\mathfrak{A}$ consisting of those elements of A supported by $\{i\}$. For $u \in U$, let $\bar{u}^i=\{f : f_i=u\}$. Now A^i is invariant under Φ and $\{\bar{u}^i : u \in U\}$ is the set of atoms of A^i; hence Φ induces a permutation of the $\bar{u}^i$'s for each fixed i. Define γ a permutation of U by $\gamma u=v$ iff $\Phi\bar{u}^i=\bar{v}^i$. This definition is independent of i, for if $i, j \in I$ and $\Phi\bar{u}^i=\bar{v}^i$, we have $\Phi\bar{u}^j=\Phi S(i,j)\bar{u}^i=S(i,j)\Phi\bar{u}^i=S(i,j)\bar{v}^i=\bar{v}^j$. We claim $\hat{\gamma} = \Phi$. Obviously $\hat{\gamma}\bar{u}^i=\Phi\bar{u}^i$ for each $u \in U$ and $i \in I$. Now for $X \in A$, let J be a support of X with $|J|<\mathfrak{M}$. Then it is easily seen that $X=\bigcup_{f\in X}\bigcap_{i\in J}\bar{f}_i^i$ and that $\bar{f}_i^i \in A$. Thus since automorphisms preserve all unions and intersections and $\hat{\gamma}$, Φ agree on the $\bar{f}_i^i$, we have $\hat{\gamma}X=\Phi X$.

The final concept of this section is the polyadic version of a concept of cylindric algebra (see [12]).

DEFINITION 1.11. *Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ and $J \subseteq I$. For $\sigma \in {}^JJ$, let $\sigma^+=\sigma \cup \delta_{I\sim J}$ and $S_J(\sigma)=S(\sigma^+)$. Then the algebra $\boldsymbol{Rd}_J\mathfrak{A}=\langle A, +, \cdot, -, 0, 1, S_J(\sigma), \exists(K)\rangle_{\sigma\in{}^JJ, K\subseteq J}$ is called the **J-reduct** of $\mathfrak{A}$.*

THEOREM 1.12. *For $\mathfrak{A}$ a $\boldsymbol{PA}_I$ and $J \subseteq I$, $\boldsymbol{Rd}_J\,\mathfrak{A}$ is a $\boldsymbol{PA}_J$. If J is infinite, the local degree of $\boldsymbol{Rd}_J\,\mathfrak{A}$ is the minimum of $|J|^+$ and the local degree of $\mathfrak{A}$.*

$\boldsymbol{Rd}_J\,\mathfrak{A}$ is called "*the algebra obtained by fixing the variables $I\sim J$ of $\mathfrak{A}$*".

2. **Dilations and compressions.** Most of the concepts and results in this section are taken from Daigneault-Monk [5] and proofs can be found there. We shall indicate a proof for those results which are not proved in that paper.

DEFINITION 2.1. *Let $\mathfrak{A}$ be a $\boldsymbol{PA}_I$ and I^+ be a superset of I. An **I^+-dilation** of $\mathfrak{A}$ is a $\boldsymbol{PA}_{I^+}$ $\mathfrak{B}=\langle B, +, \cdot, -, 0, 1, S^+(\sigma), \exists^+(J)\rangle$ such that*

(i) *$\mathfrak{A}$ is a Boolean subalgebra of $\mathfrak{B}$;*

(ii) *for $x \in A$ and $J \subseteq I$, $\exists(J)x=\exists^+(J)x$;*

(iii) *for $x \in A$ and $\sigma \in {}^II$, $S(\sigma)x=S^+(\sigma^+)x$ where $\sigma^+=\sigma \cup \delta_{I^+\sim I}$;*

(iv) *for $x \in A$, $\sigma, \tau \in {}^{I^+}I^+$, if $\sigma\restriction I=\tau\restriction I$, then $S^+(\sigma)x=S^+(\tau)x$.*

$\mathfrak{B}$ is a minimal I^+-dilation if there is no I^+-dilation $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{C} \subset \mathfrak{B}$.

THEOREM 2.2. *For I infinite and I^+ any superset of I, every $\boldsymbol{PA}_I$ has one and to within isomorphism only one minimal I^+-dilation.*

For the proof of 2.2 it is actually necessary to prove some of the properties of minimal I^+-dilations which follow here. Some of these properties are crucial to our later work.

THEOREM 2.3. *Suppose I is an infinite set, I^+ a superset of I, $\mathfrak{A}$ a $\boldsymbol{PA}_I$, and $\mathfrak{B}$ a minimal I^+-dilation of $\mathfrak{A}$. Then*:

(i) *every element of $\mathfrak{B}$ can be written in the form $S^+(\sigma)x$ where $x \in A$, $\sigma \in {}^{I^+}I^+$ and $\sigma \restriction I$ is one to one*;

(ii) *for $x \in A$, if $J \subseteq I$ supports x in $\mathfrak{A}$, then J supports x in $\mathfrak{B}$; in particular, I supports every element of $\mathfrak{A}$ in $\mathfrak{B}$*;

(iii) *the local degrees of $\mathfrak{A}$ and $\mathfrak{B}$ are equal.*

THEOREM 2.4. *Suppose I^+ is a superset of the infinite set I, $\mathfrak{A}$ and $\mathfrak{B}$ are $\boldsymbol{PA}_I$'s, f is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, $\mathfrak{A}^+$ is a minimal I^+-dilation of $\mathfrak{A}$, $\mathfrak{B}^+$ an I^+-dilation of $\mathfrak{B}$; then there is a unique homomorphism f^+ from $\mathfrak{A}^+$ to $\mathfrak{B}^+$ such that $f^+ \restriction A = f$. If $\mathfrak{B}^+$ is a minimal I^+-dilation of $\mathfrak{B}$, then f^+ is onto if and only if f is.*

Proof. By 2.3(i), every element of $\mathfrak{A}^+$ can be written $S^+(\sigma)x$ where $\sigma \in {}^{I^+}I^+$ and $x \in A$. We define $f^+S^+(\sigma)x$ to be $S^+(\sigma)fx$. Then following the methods of §§3 and 4 of [5], it is easy to show f^+ is well defined, a homomorphism, and has the required one to one and onto properties.

Next, we introduce a notion dual to dilation.

DEFINITION 2.5. *Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ and $J \subseteq I$. The* ***J-compression*** *of $\mathfrak{A}$ is the algebra $\mathfrak{A}_J = \langle A_J, +, \cdot, -, 0, 1, S_J(\sigma), \exists_J(K)\rangle$ where $A_J = \{x \in A : J \text{ supports } x\}$, $S_J(\sigma)x$ is defined for $x \in A_J$ and $\sigma \in {}^JJ$ by $S_J(\sigma)x = S(\sigma \cup \delta_{I \sim J})x$, and $\exists_J(K)x$ is defined for $K \subseteq J$ and $x \in A_J$ by $\exists_J(K)x = \exists(K)x$.*

THEOREM 2.6. *If $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ and $J \subseteq I$, then $\mathfrak{A}_J$ is a $\boldsymbol{PA}_J$; $\mathfrak{A}$ is an I-dilation of $\mathfrak{A}_J$.*

If $\mathfrak{A}$ is a minimal I-dilation of $\mathfrak{A}_J$, $\mathfrak{A}_J$ is called a *faithful* compression of $\mathfrak{A}$.

THEOREM 2.7. *Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$, $J \subseteq I \subseteq I^+$ are infinite. Then*

(i) *If $\mathfrak{B}$ is a minimal I^+ dilation of $\mathfrak{A}$, then $\mathfrak{B}_I = \mathfrak{A}$ and is a faithful I-compression of $\mathfrak{B}$.*

(ii) *$\mathfrak{A}_J$ is a faithful compression of $\mathfrak{A}$ iff $|J| \geqq \mathfrak{N}$ where $\mathfrak{N}$ is the cardinal predecessor of the local degree $\mathfrak{M}$ of $\mathfrak{A}$, i.e. $\mathfrak{N} = \mathfrak{M}$ if $\mathfrak{M}$ is a limit cardinal and $\mathfrak{N}^+ = \mathfrak{M}$ if $\mathfrak{M}$ is a successor cardinal.*

THEOREM 2.8. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are $\boldsymbol{PA}_I$'s and f is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Then $f_J = f \restriction A_J$ is a homomorphism from $\mathfrak{A}_J$ into $\mathfrak{B}_J$ and f_J is one to one if and only if f is. If J is infinite and $\mathfrak{B}_J$ is a faithful compression, then f_J is onto iff f is.*

The next theorem follows easily from 2.4 and 2.3(i).

THEOREM 2.9. *Suppose I is an infinite set, $I \subseteq I^+$, Z any set with $Z \cap I^+ = 0$ and $\mathfrak{A}$ a $\boldsymbol{PA}_I$. If $\mathfrak{B}$ is a minimal $I^+ \cup Z$-dilation of $\mathfrak{A}$, $\mathfrak{C}$ a minimal $I \cup Z$-dilation of $\mathfrak{A}$, and $\mathfrak{D}$ a minimal I^+-dilation of $\boldsymbol{Rd}_I\mathfrak{C}$, then $\mathfrak{D}$ is isomorphic to $\boldsymbol{Rd}_{I^+}\mathfrak{B}$.*

Proof. By 3.11 of [5], $\mathfrak{B}_{I\cup Z}$ is a minimal $I\cup Z$-dilation of $\mathfrak{A}$. So by 2.2 there is an isomorphism f of $\mathfrak{C}$ onto $\mathfrak{B}_{I\cup Z}$. Then f is an isomorphism from $\boldsymbol{Rd}_I\mathfrak{C}$ onto $\boldsymbol{Rd}_I\mathfrak{B}_{I\cup Z}$ is enough to show that $\boldsymbol{Rd}_{I^+}\mathfrak{B}$ is a minimal I^+-dilation of $\boldsymbol{Rd}_I\mathfrak{B}_{I\cup Z}$. Suppose $p\in B$; then p may be written $S(\sigma)q$ for some $q\in A$, $\sigma\in {}^{I^+\cup Z}(I^+\cup Z)$ with $\sigma\restriction I$ one to one. There is $\tau\in {}^{I^+\cup Z}(I^+\cup Z)$ and $\theta\in {}^{I^+}I^+$ such that $\theta\tau\restriction I=\sigma\restriction I$. Then we have $p=S(\theta)S(\tau)q$ and $S(\tau)q\in B_{I\cup Z}$. Thus p is in the minimal I^+-dilation of $\boldsymbol{Rd}_I\mathfrak{B}_{I\cup Z}$ which is contained in $\boldsymbol{Rd}_{I^+}\mathfrak{B}$. Therefore $\boldsymbol{Rd}_{I^+}\mathfrak{B}$ is a minimal I^+-dilation of $\boldsymbol{Rd}_I\mathfrak{B}_{I\cup Z}$.

All of these results apply equally well to $\boldsymbol{PEA}_I$'s. This can easily be seen from the next easily verified theorem.

THEOREM 2.10. *Suppose I is infinite, $\mathfrak{A}$ a $\boldsymbol{PEA}_I$ and I^+ a superset of I. Then there is one and up to isomorphism only one minimal I^+-dilation $\mathfrak{B}$ of $\mathfrak{A}$. (Of course $\mathfrak{B}$ is to be a $\boldsymbol{PEA}_{I^+}$.)*

3. **Constants.** Constants were first introduced in polyadic algebra by Halmos [8]. His notion proves satisfactory for the study of locally finite algebras, but when local finiteness is not required, serious technical difficulties arise. It thus becomes desirable to reformulate the notion. The first such reformulation was suggested by Donald Monk to the present author, who added one axiom schema to the definition. Our definition reduces in all essential properties to Halmos' definition when the algebra in question is locally finite. Actually, if Y is taken with a single element, our definition and Halmos' coincide.

DEFINITION 3.1. *Let $\mathfrak{A}$ be a $\boldsymbol{PA}_I$ and Y a nonempty set. A Y-constant on $\mathfrak{A}$ is a function $T(\ /\ )$ from ${}^IY\times S(I)$ into the Boolean endomorphisms of $\mathfrak{A}$ such that for all $y, y_1, y_2\in {}^IY$ and $J, K\subseteq I$, $\sigma\in {}^II$*

(C1) $T(y/0)=\delta_A$;

(C2) $T(y/J\cup K)=T(y/J)T(y/K)$;

(C3) $T(y/J)\,\exists(K)=\exists(K)T(y/J\sim K)$;

(C4) $\exists(J)T(y/K)=T(y/K)\,\exists(J\sim K)$;

(C5) $T(y/J)S(\sigma)=S(\sigma)T(y\sigma/\sigma^{-1}J)$;

(C6) *if $y_1\restriction J=y_2\restriction J$, then $T(y_1/J)=T(y_2/J)$.*

It has been pointed out by Monk that when $|I|\geqq 2$, (C3) is redundant. Of course if $|I|\leqq 1$, (C5) and (C6) are redundant, so the problem remains to find an independent axiom system which works for all I.

We give two natural examples of this notion.

EXAMPLE 1. Let $\mathfrak{A}$ be a $\boldsymbol{PSA}_I$ with base U and $0\neq Y\subseteq U$. For $y\in {}^IY$, $J\subseteq I$, and $X\in A$, define $T(y/J)X=\{x\in {}^IU:$ there is $z\in X$ with $z\restriction I\sim J=x\restriction I\sim J$ and $z\restriction J=y\restriction J\}$. Then T is a Y-constant on U provided for every $X\in A$, $y\in {}^IY$, and $J\subseteq I$, that $T(y/J)X\in A$. T is called a *functional constant*. It is unknown if every constant on a $\boldsymbol{PSA}_I$ is essentially equivalent to a functional constant.

EXAMPLE 2. Suppose $Y\cap I=0$ and $\mathfrak{A}=\boldsymbol{Rd}_I\mathfrak{B}$ where $\mathfrak{B}$ is a $\boldsymbol{PA}_{I\cup Y}$. For $p\in A$, $y\in {}^IY$, $J\subseteq I$, define $T(y/J)p=S^{\mathfrak{B}}(\bar{y}^J)p$ where $\bar{y}^J=y\restriction J\cup\delta_{I\cup Y\sim J}$. Then T is a Y-constant on $\mathfrak{A}$ and T is said to be obtained by fixing the variables Y of $\mathfrak{B}$. $\mathfrak{A}$ is

said to be the algebra obtained by fixing the variables Y of $\mathfrak{B}$. Constants obtained in this way play an important role in later developments.

LEMMA 3.2. *For all $p \in A$, J, $K \subseteq I$, y, $z \in {}^I Y$*

(i) *$T(y/J)p$ is independent of J;*

(ii) *if K supports p, then $T(y/J)p = T(y/J \cap K)p$;*

(iii) *$T(y/J)T(z/K)p = T(y/J \sim K)T(z/K)p$;*

(iv) *$\forall(J)p \leqq T(y/J)p \leqq \exists(J)p$.*

Proof.

(i)
$$\begin{aligned} \exists(J)T(y/J)p &= T(y/J)\exists(0)p \qquad \text{by C4} \\ &= T(y/J)p \end{aligned}$$

(ii)
$$\begin{aligned} T(y/J)p &= T(y/J)\exists(I \sim K)p \\ &= \exists(I \sim K)T(y/J \sim (I \sim K))p \\ &= \exists(I \sim K)T(y/J \cap K)p \\ &= T(y/J \cap K)\exists(I \sim K)p \\ &= T(y/J \cap K)p. \end{aligned}$$

(iii) and (iv) follow from (i) and (ii).

LEMMA 3.3. *Suppose T is Y-constant on $\mathfrak{A}$ and f is a homomorphism from $\mathfrak{A}$ **onto** $\mathfrak{B}$. Then T' defined on $\mathfrak{B}$ for $y \in {}^I Y$, $J \subseteq I$, and $p \in A$ by $T'(y/J)fp = fT(y/J)p$ is a Y-constant on $\mathfrak{B}$.*

Proof. Due to the equational nature of the definition of Y-constant, it is sufficient to show that T' is well defined. Suppose p, $q \in A$ and $fp = fg$; then $f(p \oplus q) = 0$, whence $f\exists(J)(p \oplus q) = 0$. We then have

$$\begin{aligned} fT(y/J)p \oplus fT(y/J)q &= f(T(y/J)p \oplus T(y/J)q) \\ &= fT(y/J)(p \oplus q) \\ &\leqq f\exists(J)(p \oplus q) \qquad \text{by 3.2(iv)} \\ &= 0. \end{aligned}$$

Here $p \oplus q = p \cdot -q + q \cdot -p$, the symmetric difference of p and q.

LEMMA 3.4. *Suppose x, y, $z \in {}^I Y$, J, $K \subseteq I$, $x \restriction J = z \restriction J$, and $y \restriction K = z \restriction K$; then $T(x/J)T(y/k) = T(z/J \cup K)$.*

Proof. By (C6).

The preceding lemmas will constantly be used without citation. The next lemma is very useful in constructing homomorphisms.

LEMMA 3.5. *Let I be infinite and $\mathfrak{A}$ be a $\boldsymbol{PA}_I$ with local degree $\mathfrak{M} \leqq |I|$ and Y-constant T. Suppose p, $q \in A$, J, $K \subseteq I$, J', K' are supports of p, q respectively with*

$|J'|$, $|K'|<\mathfrak{M}$, and $x, y \in {}^I Y$. Further suppose σ, τ are permutations of I and $z \in {}^I Y$ satisfying $\sigma\restriction J' \sim J = \delta_{J'\sim J}$, $\tau\restriction K' \sim K = \delta_{K'\sim K}$, the three sets $J' \cup K'$, $\sigma(J \cap J')$, $\tau(K \cap K')$ are pairwise disjoint, and $z\sigma\restriction J \cap J' = x\restriction J \cap J'$, $z\tau\restriction K \cap K' = y\restriction K \cap K'$. Then $T(z/\sigma(J \cap J') \cup \tau(K \cap K'))(S(\sigma)p + S(\tau)q) = T(x/J)p + T(y/K)q$.

Proof. Since $T(\ /\ )$ is a Boolean homomorphism, we shall only show

$$T(z/\sigma(J \cap J') \cup \tau(K \cap K'))S(\sigma)p = T(x/J)p$$

(the other half involving q is similar).

$$\begin{aligned} T(z/\sigma(J \cap J') \cup \tau(K \cap K'))S(\sigma)p &= T(z/\sigma(J \cup J'))S(\sigma)p \\ &= S(\sigma)T(z\sigma/\sigma^{-1}\sigma(J \cap J'))p \\ &= S(\sigma)T(y/J \cap J')p \\ &= S(\sigma)T(y/J)p \\ &= T(y/J)p \end{aligned}$$

(since this is supported by $J' \sim J$ and $\sigma\restriction J \sim J' = \delta_{J\sim J'}$).

We shall illustrate the use of 3.5 in the proof of Theorem 3.10; in other places where the application is as in 3.10, we shall only state that 3.5 is the justification of the statement to which it applies.

LEMMA 3.6. *Let $\mathfrak{A}$ be a $\boldsymbol{PA}_I$ with Y-constant T. Take $y_0 \in Y$ and suppose $J \subseteq I$. Then T^- defined for the J-compression $\mathfrak{A}_J$ of $\mathfrak{A}$ or on $\boldsymbol{Rd}_J\mathfrak{A}$ by $T^-(y/K) = T(y^+/K)$ for $K \subseteq J$ and $y \in {}^J Y$ is a Y-constant. Here $y^+ \in {}^I Y$ is defined by $y_i^+ = y_i$ if $i \in J$ and $y_i^+ = y_0$ if $i \in I \sim J$.*

It should be noted that by (C6), there is really no necessity to take a fixed y_0 in 3.6. It would be sufficient to let y^+ be any extension of y to a function from I into Y. (C6) then says that the value of $T(y^+/K)$ is independent of the choice of y^+.

THEOREM 3.7. *Let I be infinite, $\mathfrak{A}$ a $\boldsymbol{PA}_I$ with Y-constant T, I^+ a superset of I, and $\mathfrak{B}$ a minimal I^+-dilation of $\mathfrak{A}$. There is a unique Y-constant T^+ on $\mathfrak{B}$ which extends T, i.e. for $p \in A$, $y \in {}^{I^+} Y$, and $J \subseteq I$, $T^+(y/J)p = T(y\restriction I/J)p$.*

Proof. Uniqueness. Suppose $q \in B$, $y \in {}^{I^+} Y$, $J \subseteq I^+$. By 2.3(i), $q = S^+(\sigma)p$ for some $p \in A$ and $\sigma \in {}^{I^+}I^+$. We calculate

$$\begin{aligned} T^+(y/J)q &= T^+(y/J)S^+(\sigma)p \\ &= S^+(\sigma)T^+(y\sigma/\sigma^{-1}J)p \\ &= S^+(\sigma)T^+(y\sigma/\sigma^{-1}J \cap I)p \qquad (I \text{ supports } p) \\ &= S^+(\sigma)T(y\sigma\restriction I/\sigma^{-1}J \cap I)p. \end{aligned}$$

Thus T^+ is completely determined by T.

Existence. Notice that in virtue of 3.6, we may assume that $|I| < |I^+|$; the case $|I| = |I^+|$ then follows by taking an I^+-compression of a suitably larger dilation.

Every element of $\mathfrak{B}$ has the form $S^+(\sigma)p$ where $p \in A$ and $\sigma \in {}^{I^+}I^+$. For $y \in {}^{I^+}Y$ and $J \subseteq I^+$ we define $T^+(y/J)S^+(\sigma)p = S^+(\sigma)T(y\sigma\restriction I/\sigma^{-1}J \cap I)p$. It is straightforward but tedious to show $T^+(y/J)$ is well defined and satisfies (C1)–(C6).

LEMMA 3.8. *Suppose I is infinite, $\mathfrak{A}$ a $\boldsymbol{PA}_I$ and $\mathfrak{A}$ has local degree $\mathfrak{M} \leqq |I|$. If $\mathfrak{A}'$ is the algebra obtained by fixing the variables $Y = I^+ \sim I$ of a minimal I^+-dilation of $\mathfrak{A}$, and T is the Y-constant on $\mathfrak{A}'$ so obtained; then every element of $\mathfrak{A}'$ can be written in the form $T(y/J)p$ for some $p \in A$, $J \subseteq I$, and $y \in {}^IY$. Furthermore, y can be taken one to one on J.*

Proof. Every element has the form $S^+(\sigma)p$ where $p \in A$, $\sigma \in {}^{I^+}I^+$ and $\sigma\restriction I$ is one to one. Let K be a support of p with $|K| < \mathfrak{M}$. Choose $\tau \in {}^II$ so that $\tau\restriction K$ is one to one, $\tau\restriction\sigma^{-1}(I) \cap I = \sigma\restriction\sigma^{-1}(I) \cap I$ and $\tau(K \sim \sigma^{-1}(I)) \cap (K \cup \sigma(K)) = 0$. Let $J = \tau(K \sim \sigma^{-1}(I))$ and choose $y \in {}^IY$ such that $y\tau\restriction K \sim \sigma^{-1}(I) = \sigma\restriction K \sim \sigma^{-1}I$. Then $y\restriction J$ is one to one and $S^+(\sigma)p = T(y/J)S(\tau)p$.

LEMMA 3.9. *Suppose I is infinite and T is a Y-constant on a $\boldsymbol{PA}_I$ $\mathfrak{A}$. Further suppose Z is a set disjoint from I and Y and T' is the Z-constant on $\mathfrak{A}' = \boldsymbol{Rd}_I\mathfrak{A}^+$ obtained by fixing the variables Z in a minimal $I \cup Z$-dilation $\mathfrak{A}^+$ of $\mathfrak{A}$. Further let $\tilde{T}$ be the Y-constant on $\mathfrak{A}'$ obtained via 3.7 and 3.6. Then if $y \in {}^IY$, $z \in {}^IZ$, $J, K \subseteq I$, and $J \cap K = 0$, we have $\tilde{T}(y/J)T'(z/K) = T'(z/K)\tilde{T}(y/J)$.*

Proof. In this proof, if $t \in {}^J(I \cup Z)$ for any $J \subseteq I \cup Z$, let $t^+ = t \cup \delta_{I \cup Z \sim J}$. Let $y^+ \in {}^{I \cup Z}Y$ be any extension of y, i.e. $y^+\restriction I = y$. Let T^+ be the extension of T to $\mathfrak{A}^+$ as in 3.7. Then

$$\begin{aligned}
\tilde{T}(y/J)T'(z/K) &= T^+(y^+/J)S^+((z\restriction K)^+) \\
&= S^+((z\restriction K)^+)T^+(y^+(z\restriction K)^+/(z\restriction K)^{+-1}J) \\
&= T'(z/K)T^+(y^+/J) \qquad \text{because } J \cap (K \cup Z) = 0 \\
&= T'(z/K)\tilde{T}(y/J).
\end{aligned}$$

The next theorem says in effect that if $Z \cap I = 0$ then the algebra $\mathfrak{A}'$ obtained by fixing the variables Z of a minimal $I \cup Z$-dilation of $\mathfrak{A}$ is a "free extension of $\mathfrak{A}$" by the Z-constant T. In the case that $\mathfrak{A}$ is locally finite, this theorem is due to Daigneault [4].

THEOREM 3.10. *Suppose $\mathfrak{A}$ has local degree $\mathfrak{M} \leqq |I|$ and I is infinite. Further suppose $\mathfrak{B}$ is a $\boldsymbol{PA}_I$ with Y-constant T and f is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Then if $Z \neq 0$ and t is a function from Z into Y, there is an extension of f to a homomorphism g from $\mathfrak{A}'$ into $\mathfrak{B}$, where $\mathfrak{A}'$ is the algebra obtained by fixing the variables Z in a minimal $I \cup Z$-dilation $\mathfrak{A}^+$ of $\mathfrak{A}$. Furthermore, if T' is the Z-constant on $\mathfrak{A}'$ obtained by fixing the variables Z, g may be taken so that for all $p \in A'$, $z \in {}^IZ$, and $J \subseteq I$, $gT'(z/J)p = T(tz/J)gp$.*

Proof. In this proof, if $J \subseteq Z \cup I$ and $y \in {}^J(Z \cup I)$, we define $y^+ = y \cup \delta_{I \cup Z \sim J}$. First we consider the case where $\mathfrak{M} \leqq |Z|$. By 3.8 every element $q \in A'$ can be written in the form $q = T'(y/J)p$ where $p \in A$, $y \in {}^IZ$, $J \subseteq I$, and $y \restriction J$ is one to one. Define $gq = T(ty/J)fp$. We claim g is the required homomorphism.

To show g is well defined, suppose $q = T'(y_1/J_1)p_1 = T'(y_2/J_2)p_2$ where p_1, $p_2 \in A$, y_1, $y_2 \in {}^IZ$, J_1, $J_2 \subseteq I$, and for $i = 1, 2$, $y_i \restriction J_i$ is one to one. Let K_1, K_2 be supports of p_1, p_2 with $|K_1|$, $|K_2| < \mathfrak{M}$. We may assume $|J_1| = |J_2| < \mathfrak{M}$ and $y_1(J_1) = y_2(J_2)$; otherwise choose J_1', J_2' so that $J_i \cap K_i = J_i' \cap K_i$ for $i = 1, 2$, $|(J_1' \sim K_1)| = |y_2(J_2 \cap K_2) \sim y_1(J_1 \cap K_1)|$, and $|J_2' \sim K_2| = |y_1(J_1 \cap K_1) \sim y_2(J_2 \cap K_2)|$. Notice $|J_1'| = |y_1(J_1 \cap K_1) \cup y_2(J_2 \cap K_2)| = |J_2'| < \mathfrak{M}$. Take y_1', $y_2' \in {}^IZ$ so that $y_i' \restriction J_i \cap K_i = y_i \restriction J_i \cap K_i$ for $i = 1, 2$, $y_1'(J_1' \sim K_1) = y_2(J_2 \cap K_2) \sim y_1(J_1 \cap K_1)$, $y_2'(J_2' \sim K_2) = y_1(J_1 \cap K_1) \sim y_2(J_2 \cap K_2)$, and $y_i' \restriction J_i$ one to one for $i = 1, 2$. Then for $i = 1, 2$, $T'(j_i'/J_i')p_i = T'(y_i/J_i)p_i$ and $T(ty_i'/J_i')fp_i = T(ty_i/J_i)fp_i$. Furthermore, $y_1'(J_1') = y_1(J_1 \cap K_1) \cup y_2(J_2 \cap K_2) = y_2'(J_2')$. Thus we could replace J_i by J_i' and y_i by y_i' for $i = 1, 2$, so we do assume $|J_1| = |J_2| < \mathfrak{M}$ and $y_1(J_1) = y_2(J_2)$.

Now choose α to be a transformation of $I \cup Z$ satisfying

(1) $\alpha y_2 \restriction J_2 = \delta_{J_2}$;

(2) α is the identity outside of $y_1(J_1) = y_2(J_2)$.

Notice that αy_1 maps J_1 one to one onto J_2. Now let $\beta = \alpha(y_1 \restriction J_1)^+ \restriction I$ and we obtain

$$\begin{aligned} p_2 &= S^+(\alpha)S^+((y_2 \restriction J_2)^+)p_2 \qquad (\text{since } \alpha(y_2 \restriction J_2)^+ \restriction I = \delta_I) \\ &= S^+(\alpha)S^+((y_1 \restriction J_1)^+)p_1 \\ &= S^+(\alpha(y_1 \restriction J_1)^+)p_1 \\ &= S(\beta)p_1. \end{aligned}$$

Next observe that $y_2\beta \restriction J_1 = y_1 \restriction J_1$ and that q is supported in A^+ by $I \cup Z \sim J_1$. Thus p_2 is supported in $\mathfrak{A}$ by $I \cap \alpha(I \cup Z \sim J_1)$ and this last set is disjoint from $J_1 \sim J_2$. Hence p_2 is independent of $J_1 \sim J_2$. By the symmetry of the situation we may also conclude that p_1 is independent of $J_2 \sim J_1$. Now we calculate

$$\begin{aligned} T(ty_2/J_2)fp_2 &= T(ty_2/J_2)fS(\beta)p_1 \\ &= T(ty_2/J_2)S(\beta)fp_1 \\ &= S(\beta)T(ty_2\beta/J_1 \cup J_2)fp_1 \\ &= T(ty_2\beta/J_1)fp_1 \\ &= T(ty_1/J_1)fp_1. \end{aligned}$$

Thus g is well defined.

Next we show for $q \in A$; $y \in {}^IZ$ and $J \subseteq I$ that $gT'(y/J)q = T(ty/J)gq$. For this suppose $q = T'(z/K)p$ where $p \in A$, $z \in {}^IZ$, $J \subseteq I$ and $z \restriction J$ is one to one. Let $x \in {}^IZ$ with $x \restriction K = z \restriction K$ and $x \restriction J \sim K = y \restriction J \sim K$. Then $T'(y/J)q = T'(x/J \cup K)p$. Now let L be a subset of $J \cup K$ so that $x \restriction L$ is one to one and $x(L) = x(J \cup K)$. Take $\sigma \in {}^II$

with $\sigma(J \cup K)=L$, $x\sigma\restriction J \cup K = x\restriction J \cup K$, and $\sigma\restriction I \sim (J \cup K) = \delta_{I\sim(J\cup K)}$. Then $T'(x/L)S(\sigma)p = S(\sigma)T'(x\sigma/J \cup K)p = T'(x/J \cup K)p$. Thus we have

$$\begin{aligned} gT'(y/J)q &= gT'(x/J \cup K)p \\ &= gT'(x/L)S(\sigma)p \\ &= T(tx/L)fS(\sigma)p \qquad \text{by the definition of } g \\ &= T(tx/L)S(\sigma)fp \\ &= S(\sigma)T(tx/J \cup K)fp \\ &= T(ty/J)T(tz/K)fp \\ &= T(ty/J)gq. \end{aligned}$$

Next we show g preserves the Boolean operations. It is clear that g preserves $-$. To see g preserves $+$, suppose $q_1, q_2 \in A'$, $q_i = T'(y_i/J_i)p_i$ where $p_i \in A$, $y_i \in {}^IZ$, $J_i \subseteq I$ and $|J_i| < \mathfrak{M}$ for $i=1, 2$. Then let K_1, K_2 be supports of p_1, p_2 with $|K_1|$, $|K_2| < \mathfrak{M}$ and $J_i \subseteq K_i$ for $i=1, 2$. Choose for $i=1, 2$ permutations σ_i of I such that $\sigma_i K_i \sim J_i = \delta_{K_i \sim J_i}$ and $K_1 \cup K_2$, $\sigma_1(J_1)$, $\sigma_2(J_2)$ are pairwise disjoint. Finally take $y \in {}^IZ$ with $y\sigma_i\restriction J_i = y_i\restriction J_i$ for $i=1, 2$ and $J = \sigma_1(J_1) \cup \sigma_2(J_2)$. Then

$$\begin{aligned} g(q_1+q_2) &= g(T'(y_1/J_1)p_1 + T'(y_2/J_2)p_2) \\ &= gT'(y/J)(S(\sigma_1)p_1 + S(\sigma_2)p_2) \qquad \text{(by 3.5)} \\ &= T(ty/J)f(S(\sigma_1)p_1 + S(\sigma_2)p_2) \\ &= T(ty/J)(S(\sigma_1)fp_1 + S(\sigma_2)fp_2) \\ &= T(ty_1/J_1)fp_1 + T(ty_2/J_2)fp_2 \qquad \text{(by 3.5)} \\ &= gq_1 + gq_2. \end{aligned}$$

To see g preserves $S(\tau)$ suppose $q = T'(y/J)p$ where $p \in A$, $y \in {}^IZ$, $J \subseteq I$, $|J| < \mathfrak{M}$, and $y\restriction J$ is one to one. Let $K \subseteq I$ such that $K \cap J = 0$, $K \cup J$ supports p, and $|K| < \mathfrak{M}$. Notice that then K supports q. Take $\sigma \in {}^II$ so that $\sigma\restriction K = \tau\restriction K$ and $\sigma\restriction I \sim K = \delta_{I\sim K}$. Choose α a permutation of I so that $\alpha\restriction K = \delta_K$ and $\alpha(J) \cap (K \cup \sigma(K)) = 0$. Then it is easy to see that $T'(y\alpha^{-1}/\alpha J)S(\alpha)p = T'(y/J)p$. Hence we may assume that $J \cap (K \cup \sigma(K)) = 0$ (else replace p by $S(\alpha)p$, J by $\alpha(J)$ and y by $y\alpha^{-1}$). Then

$$\begin{aligned} gS(\tau)q &= gS(\sigma)T'(y/J)p \\ &= gT'(y/J)S(\sigma)p \qquad (\sigma(J) = J \text{ and } y\sigma\restriction J = y\restriction J) \\ &= T(ty/J)fS(\sigma)p \\ &= T(ty/J)S(\sigma)fp \\ &= S(\sigma)T(ty/J)fp \\ &= S(\sigma)gq \\ &= S(\tau)gq. \end{aligned}$$

To see g preserves $\exists(K)$, assume $q=T'(y/J)p$ with $p\in A$, $y\in {}^IZ$, $J\subseteq I$. Then

$$\begin{aligned} g\,\exists(K)q &= g\,\exists(K)T'(y/J)p \\ &= gT'(y/J)\,\exists(K\sim J)p \\ &= T(ty/J)f\,\exists(K\sim J)p \\ &= T(ty/J)\,\exists(K\sim J)fp \\ &= \exists(K)T(ty/J)fp \\ &= \exists(K)q. \end{aligned}$$

This completes the proof for the case $|Z|\geqq\mathfrak{M}$. If $|Z|<\mathfrak{M}$, take $Z'\supseteq Z$ with $Z'\cap I=0$ and $|Z'|=\mathfrak{M}$. Let t' be a function from Z' to Y with $t'\restriction Z=t$ and let $\mathfrak{A}^*$ be the algebra obtained by fixing the variables Z' in a minimal $I\cup Z'$-dilation of $\mathfrak{A}^+$. Then if T^* is the Z'-constant on $\mathfrak{A}^*$ there is by the above a homomorphism $h\colon \mathfrak{A}^*\to\mathfrak{B}$ satisfying $hT^*(y/J)q=T(ty/J)hq$. Now $\mathfrak{A}'\subseteq\mathfrak{A}^*$ and for $q\in A'$, $y\in {}^IZ$ and $J\subseteq I$, $T'(y/J)q=T^*(y/J)q$. Thus $q=h\restriction A'$ is the required homomorphism. This completes the proof.

Theorem 3.10 and several of the subsequent results remain true when the local degree condition is removed. In each case the more general result follows by an application of 3.6 and 3.7. We will not go into the details here, but the interested reader will have no difficulty in verifying this.

If an algebra has more than one constant, it is natural to ask whether they can be put together into one large constant. Our next two lemmas show that indeed they can.

LEMMA 3.11. *Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ with local degree $\mathfrak{M}\leqq|I|$, and, for $i=1,2$, T_i is a Y_i-constant on $\mathfrak{A}$. Then if $y_i\in {}^IY_i$ and $J_i\subseteq I$, for $i=1,2$, and $J_1\cap J_2=0$, we have $T_1(y_1/J_1)T_2(y_2/J_2)=T_2(y_2/J_2)T_1(y_1/J_1)$.*

Proof. Without loss of generality, we may assume $Y_1\cap I=0$ and $Y_1\cap Y_2=0$. Let $\mathfrak{A}'$ be the algebra obtained from $\mathfrak{A}$ by fixing the variables Y_1 in a minimal $I\cup Y_1$-dilation of $\mathfrak{A}$ and T_1' be the Y_1-constant on A' obtained in this way. Let T_2' be the Y_2-constant of $\mathfrak{A}'$ obtained by extending T_2 via 3.6 and 3.7. Let g be the homomorphism from $\mathfrak{A}'$ onto $\mathfrak{A}$ obtained from the identity maps on A and Y_1 as in 3.10. Notice that for $p\in A'$ we have $T_2(y_2/J_2)gp=gT_2'(y_2/J_2)p$ since T_2' induces a Y_2-constant on $\mathfrak{A}$ via g and this agrees with T_2 on $\mathfrak{A}$. More specifically, let T_2^* be the Y_2-constant on $\mathfrak{A}$ induced by T_2' via g. Then we have (using the notation of 3.6, 3.7)

$$\begin{aligned} gT_2'(y_2/J)p &= T_2^*(y_2/J)gp \\ &= T_2^*(y_2/J)ggp \qquad \text{(notice } g \text{ is idempotent)} \\ &= gT_2'(y_2/J)gp \\ &= gT_2'(y_2^+/J)gp \\ &= gT_2(y_2/J)gp \qquad \text{by 3.7, since } gp\in A \\ &= T_2(y_2/J)gp. \end{aligned}$$

The last step holds because $T_2(y_2/J)gp \in A$ and $g \restriction A = \delta_A$. Thus for $p \in A$ we have

$$\begin{aligned} T_1(y_1/J_1)T_2(y_2/J_2)p &= T_1(y_1/J_1)T_2(y_2/J_2)gp \\ &= gT_1'(y_1/J_1)T_2'(y_2/J_2)p \\ &= gT_2'(y_2/J_2)T_1'(y_1/J_1)p \qquad \text{by } 3.9 \\ &= T_2(y_2/J_2)T_1(y_1/J_1)gp \\ &= T_2(y_2/J_2)T_1(y_1/J_1)p. \end{aligned}$$

This completes the proof.

Suppose for $i=1, 2$ that T_i is a Y_i-constant on a $\boldsymbol{PA}_I$ $\mathfrak{A}$ and $Y_1 \cap Y_2 = 0$. Let $Y = Y_1 \cup Y_2$ and choose $u_1 \in Y_1$ and $u_2 \in Y_2$. For $y \in {}^I Y$ let $y_i \in {}^I Y_i$ be defined by $y_i(k) = y(k)$ if $y_i(k) \in Y_i$ and $y_i(k) = u_i$ otherwise. Define

$$T(y/J) = T_1(y_1/J \cap y_1^{-1}(Y_1))T_2(y_2/J \cap y_2^{-1}(Y_2))$$

for $y \in {}^I Y$ and $J \subseteq I$. We then have

LEMMA 3.12. *If $\mathfrak{A}$ has local degree $\mathfrak{M} \leqq |I|$, then T is a Y-constant on $\mathfrak{A}$.*

Proof. It is clear that $T(y/J)$ is a Boolean endomorphism. All the axioms for a constant except (C2) follow by mechanical application of the corresponding axioms to T_1 and T_2. (C2) follows easily from 3.11.

Suppose T is a Y-constant on $\mathfrak{A}$ and $0 \neq Y' \subseteq Y$, then $T' = T \restriction {}^I Y' \times S(I)$ is a Y'-constant. This suggests the next definition.

DEFINITION 3.13. *If $Y' \subseteq Y$, T' is a Y'-constant and T a Y-constant on $\mathfrak{A}$ and $T' = T \restriction {}^I Y' \cap S(I)$, then T' is called the **Y'-reduction** of T and T is called a **Y-expansion** of T'.*

Notice that 3.12 is just the statement that if $Y_1 \cap Y_2 = 0$, then a Y_1 and a Y_2 constant on $\mathfrak{A}$ have a common $Y_1 \cup Y_2$ expansion.

Suppose we have an ascending chain of algebras each of which has a constant which is an expansion of the constants on the preceding algebras. It would be desirable to know that there is a constant on the union of the algebras which is an expansion of all of the constants. It seems unlikely that this is the case, though we do not have a counterexample. We do however have two special results which show the situation is almost this nice. For the remainder of this section suppose η is a fixed ordinal and for each $\alpha < \eta$, $\mathfrak{A}_\alpha$ is a $\boldsymbol{PA}_I$, T_α is a Y_α-constant on $\mathfrak{A}_\alpha$, and whenever $\beta \leqq \alpha$, $\mathfrak{A}_\beta \subseteq \mathfrak{A}_\alpha$, and T_β is the restriction to $\mathfrak{A}_\beta$ of the Y_β-reduction of T_α. Further suppose $\mathfrak{A} = \bigcup_{\alpha<\eta} \mathfrak{A}_\alpha$ and $Y = \bigcup_{\alpha<\eta} Y_\alpha$.

LEMMA 3.14. *If each $\mathfrak{A}$ has local degree $\mathfrak{M}$ and η is a regular cardinal with $\mathfrak{M} \leqq \eta$, then there is a Y-constant T on $\mathfrak{A}$ such that for each $\alpha < \eta$, T_α is the restriction to $\mathfrak{A}_\alpha$ of the Y_α-reduction of T.*

Proof. For $p \in A$, $y \in {}^I Y$ and $J \subseteq I$ we wish to define $T(y/J)p$. Let $K \subseteq I$ be a support of p with power less than $\mathfrak{M}$. There is $\beta < \eta$ such that $p \in \mathfrak{A}_\beta$ and for each

$i \in K$, $y_i \in Y_\beta$ (this is where the regularity of η is required). Take $\bar{y} \in {}^I Y_\beta$ with $\bar{y} \restriction K = y \restriction K$ and define $T(y/J)p = T_\beta(\bar{y}/J)p$. This is obviously independent of the choices of K, β, and $\bar{y}$ (so long as β is sufficiently large). It is then a straightforward matter to show T has the required properties.

LEMMA 3.15. *Suppose $\mathfrak{A}$ has local degree $\mathfrak{M} \leqq |I|$. Then there is a $\boldsymbol{PA}_I$ $\mathfrak{A}^*$ with Y-constant T such that $\mathfrak{A} \subseteq \mathfrak{A}^*$ and for each $\alpha < \eta$, T_α is the restriction to $\mathfrak{A}_\alpha$ of the Y_α-reduction of T.*

Proof. We assume without loss of generality that $Y \cap I = 0$. Let $\mathfrak{A}'$ be the algebra obtained by fixing the variables Y in a minimal $I \cup Y$-dilation $\mathfrak{A}^+$ of $\mathfrak{A}'$. Let T' be the Y-constant on $\mathfrak{A}'$ obtained by fixing the variables Y. Let M be the ideal of A' generated by $\{T'(y/J)p$: for some $\alpha < \eta$, $p \in A_\alpha$, $y \in {}^I Y_\alpha$, $J \subseteq I$, and $T_\alpha(y/J)p = 0\}$.

We will show $M \cap A = \{0\}$, but first suppose we already have this; we then proceed as follows. Take $\mathfrak{A}^* = \mathfrak{A}'/M$ and let T be the Y-constant on $\mathfrak{A}^*$ induced by T'. We may assume $\mathfrak{A} \subseteq \mathfrak{A}^*$ since $p \to p/M$ is a monomorphism from $\mathfrak{A}$ into $\mathfrak{A}^*$. Then we need that $T(y/J)p = T_\alpha(y/J)p$ whenever $p \in A$, $y \in {}^I Y_\alpha$, and $J \subseteq I$. But this is equivalent to showing $T'(y/J)p \oplus T_\alpha(y/J)p \in M$, and this follows from the fact that $T_\alpha(y/J)(p \oplus T_\alpha(y/J)p) = T_\alpha(y/J)p \oplus T_\alpha(y/J)p = 0$.

Now to show $A \cap M = \{0\}$, we first show the generators of M are closed under $\exists(I)$. Suppose $T_\alpha(y/J)p = 0$; then $0 = \exists(I)T_\alpha(y/J)p = T_\alpha(y/J)\exists(I \sim J)p$. Hence $\exists(I)T'(y/J)p = T'(y/J)\exists(I \sim J)p$ is among the generators of M if $T'(y/J)p$ is. By an application of 3.5, the generators of M are closed under $+$. Thus M consists of those elements of A' which are less than or equal to some generator of M.

Now assume $q \in M \cap A$ and $\exists(I)q = q$. Then $q \leqq T'(y/J)p$ where for some α, $p \in A_\alpha$, $y \in {}^I Y_\alpha$, and $T_\alpha(y/J)p = 0$. We may assume $y \restriction J$ is one to one, for if not, take $J' \subseteq J$ with $y \restriction J'$ one to one and $y(J') = y(J)$, and take $\sigma \in {}^I I$ so that $\sigma \restriction I \sim J = \delta_{I \sim J}$, $\sigma J = J'$, and $y\sigma \restriction I = y \restriction J$. In this case $T_\alpha(y/J')S(\sigma)p = T_\alpha(y/J)p$ and similarly with T'. So we would replace p by $S(\sigma)p$ and J by J'. Now let α be a transformation of $I \cup Y$ such that $\alpha \restriction I \cup Y \sim y(J) = \delta_{I \cup Y \sim y(J)}$ and $\alpha y \restriction J = \delta_J$. $S^+(\alpha)$ is a Boolean endomorphism such that $S^+(\alpha)T'(y/J)p = p$ and $S^+(\alpha)q = q$. Thus we have $q = S^+(\alpha)q \leqq S^+(\alpha)T'(y/J)p = p$. Now $\forall(I)q = q$ so $q \leqq \forall(I)p \leqq T_\alpha(y/J)p = 0$. This completes the proof.

The method of proof used in 3.15 is adapted from a proof in Halmos [8]. It will be used in a slightly different form in the next section.

4. **Rich algebras and representation.** For locally finite algebras the notion of richness was introduced by Halmos [8] in order to prove that all locally finite polyadic algebras of infinite degree are representable. Here we generalize Halmos work so as to remove the restriction of local finiteness. The general representation theorem is already known [5], but in the next section we need a relationship between constants and bases of the $\boldsymbol{PSA}_I$'s involved in the representations of the algebras. Throughout this section we assume I is a fixed but arbitrary infinite set.

DEFINITION 4.1. *Let $\mathfrak{A}$ be a* $\boldsymbol{PA_I}$.

(i) *If T is a Y-constant of $\mathfrak{A}$, $p \in A$, and $y \in {}^I Y$, y is a* ***witness*** *to p if $T(y/I)p = \exists (I)p$.*

(ii) *A* $\boldsymbol{PA_I}$ *$\mathfrak{B}$ is a* ***rich extension*** *of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$, there is a Y-constant T on $\mathfrak{B}$, and every $p \in A$ has a witness $y \in {}^I Y$.*

(iii) *$\mathfrak{A}$ is* ***rich*** *if $\mathfrak{A}$ is a rich extension of itself.*

LEMMA 4.2. *Suppose $\mathfrak{A}$ is a* $\boldsymbol{PA_I}$ *with Y-constant T; then there is a rich extension $\mathfrak{A}^*$ of $\mathfrak{A}$ such that T can be extended to a Y-constant of $\mathfrak{A}^*$. $\mathfrak{A}^*$ may be taken with the same local degree as $\mathfrak{A}$.*

Proof. For each $p \in A$, let I_p be a set disjoint from I and t_p a one to one function from I onto I_p. Further assume that if $p \neq q$, then $I_p \cap I_q = 0$. Let Z be the union of the I_p's and $I^+ = I \cup Z$. Take $\mathfrak{A}^+$ to be a minimal I^+-dilation of $\mathfrak{A}$ and let $\mathfrak{A}'$ be the algebra obtained by fixing the variables Z of $\mathfrak{A}^+$. Further let T' be the natural Z-constant of $\mathfrak{A}'$ (cf. §3, Example 2). Let M be the ideal of $\mathfrak{A}'$ generated by

$$\{\exists (I)p \cdot -T'(t_p/I)p : p \in A\}.$$

We claim $A \cap M = \{0\}$. Before showing this, suppose that it is so. Then we proceed as follows. Take $\mathfrak{A}^* = \mathfrak{A}/M$ and let T^* be the Z-constant on $\mathfrak{A}^*$ induced by T. We may assume $\mathfrak{A} \subseteq \mathfrak{A}^*$ because $p \mapsto p/M$ is a monomorphism. Now for $p \in A$, $T'(t_p/I)p \leqq \exists (I)p$, so $T'(t_p/I)p \oplus \exists (I)p \in M$; thus $\exists (I)p = T^*(t_p/I)p$, i.e., t_p is a witness to p in $\mathfrak{A}^*$. Finally, the Y-constant T extends to $\mathfrak{A}^*$ by 3.3, 3.6, and 3.7.

Now to show $A \cap M = \{0\}$, suppose $q \in A \cap M$, say

$$q \leqq \sum_{0 \leqq i \leqq m} \exists (I)p_i \cdot -T'(t_{p_i}/I)p_i.$$

Now, for $0 \leqq i \leqq m$, let $\sigma_i = t_{p_i} \cup t_{p_i}^{-1} \cup \delta_{Z \sim I_{p_i}}$. Then $T(t_{p_i}/I)p_i = S^+(t_{p_i} \cup \delta_z)p_i = S^+(\sigma_i)p_i$. Hence applying $\forall(Z)$ to the above inequality and using the dual of 1.1(ii), we obtain

$$\begin{aligned} q &\leqq \sum_{0 \leqq i \leqq m} \exists (I)p_i \cdot \forall(I_{p_i})S^+(\sigma_i)(-p_i) \\ &= \sum_{0 \leqq i \leqq m} \exists (I)p_i \cdot S^+(\sigma_i)\forall(I)(-p_i) \\ &= \sum_{0 \leqq i \leqq m} \exists (I)p_i \cdot -\exists (I)p_i \\ &= 0 \end{aligned}$$

as desired. This completes the proof.

Notice by 3.11, that if $\mathfrak{A}$ has Y-constant T, Y^* can then be taken with $Y \subseteq Y^*$ and T the Y-reduction of T^* restricted to $\mathfrak{A}$.

LEMMA 4.3. *A minimal I^+-dilation of a rich* $\boldsymbol{PA_I}$ *is rich. A compression of rich* $\boldsymbol{PA_I}$ *is rich.*

Proof. Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ with Y-constant T and every element of $\mathfrak{A}$ has a witness in ${}^I Y$. Further suppose $\mathfrak{A}^+$ is a minimal I^+-dilation of $\mathfrak{A}$ and T^+ is the extension of T to $\mathfrak{A}^+$. We assume here that $|I|<|I^+|$; the case $|I|=|I^+|$ follows from this case and the second part. Then for $q \in A^+$, $q=S^+(\sigma)p$ for some $p \in A$, $\sigma \in {}^{I^+}I^+$, σ a permutation of I^+. Let y be a witness to p and choose $z \in {}^{I^+}Y$ such that $z\sigma \restriction I=y$. Then z is a witness to q. Next suppose $J \subseteq I$ and $p \in A_J$. Let $y \in {}^I Y$ be a witness (in $\mathfrak{A}$) to p. Then $y \restriction J$ is a witness in $\mathfrak{A}_J$ to p. Here we have in mind the Y-constant T^- on $\mathfrak{A}_J$ as described in 3.6.

THEOREM 4.4. *If $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ of local degree $\mathfrak{M}$ with Y-constant T, then $\mathfrak{A}$ can be embedded in a rich $\boldsymbol{PA}_I$ $\mathfrak{A}^*$ of local degree $\mathfrak{M}$. In fact, $\mathfrak{A}$ can be embedded in a $\boldsymbol{PA}_I$ $\mathfrak{A}^*$ with Y^*-constant T^* such that every element of $\mathfrak{A}^*$ has a witness in ${}^I Y^*$, $Y \subseteq Y^*$ and T is the restriction to $\mathfrak{A}$ of the Y-reduction of T^*.*

Proof. By 4.3 we can assume that $\mathfrak{M} \leqq |I|$. Let $\mathfrak{N}$ be the smallest regular cardinal at least as large as $\mathfrak{M}$. For each $\alpha<\mathfrak{N}$, we define $\mathfrak{A}_\alpha$ with Y_α-constant T_α by transfinite recursion. $\mathfrak{A}_0=\mathfrak{A}$, $Y_0=Y$, and $T_0=T$. $\mathfrak{A}_{\alpha+1}$ is a rich extension of $\mathfrak{A}_\alpha$ having $Y_{\alpha+1}$-constant $T_{\alpha+1}$ such that every $p \in A_\alpha$ has a witness in ${}^I Y_{\alpha+1}$, $Y_\alpha \subseteq Y_{\alpha+1}$, and T_α is the restriction to $\mathfrak{A}_\alpha$ of the Y_α-reduction of $T_{\alpha+1}$. Furthermore, $\mathfrak{A}_{\alpha+1}$ has local degree $\mathfrak{M}$. This is possible because of 4.2. For α a limit ordinal, we take $\mathfrak{A}_\alpha$ to be an algebra containing $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta$ and having Y_α-constant T_α such that $Y_\alpha=\bigcup_{\beta<\alpha} Y_\beta$ and for $\beta<\alpha$, T_β is the restriction to $\mathfrak{A}_\beta$ of the Y_β-reduction of T_α. This is possible by 3.15. Then we take $\mathfrak{A}^*=\bigcup_{\alpha<\mathfrak{N}} \mathfrak{A}_\alpha$ and $Y^*=\bigcup_{\alpha<\mathfrak{N}} Y_\alpha$. Then by 3.14 there is a Y^*-constant T^* on $\mathfrak{A}^*$ such that for $\alpha<\mathfrak{N}$, T_α is the restriction to $\mathfrak{A}_\alpha$ of the Y_α-reduction of T^*. Further, $\mathfrak{A}^*$ is rich, for if $p \in A^*$ then $p \in A_\alpha$ for some α. Take $y \in {}^I Y_{\alpha+1}$ a witness to p; then $T^*(y/I)p=T_\alpha(y/I)p=\exists(I)p$. This completes the proof.

LEMMA 4.5. *A homomorphic image of a rich algebra is rich.*

THEOREM 4.6. *Let $\mathfrak{A}$ be a rich $\boldsymbol{PA}_I$ with Y-constant T such that every element of $\mathfrak{A}$ has a witness in ${}^I Y$. Let M be a Boolean ultrafilter on $\mathfrak{A}$. Then the function f defined by $fp=\{y \in {}^I Y : T(y/I)p \in M\}$ is a homomorphism from $\mathfrak{A}$ into the $\boldsymbol{PSA}_I$ with base Y. Furthermore, if T' is the functional Y-constant described in Example* 1 *of* §3, *then for $p \in A$, $y \in {}^I Y$, and $J \subseteq I$, $fT(y/J)fp=T'(y/J)fp$.*

The proof is similar to the proof of 11.1 in [4].

From 4.4 and 4.5 it follows that every simple $\boldsymbol{PA}_I$ is isomorphic to a $\boldsymbol{PSA}_I$. This together with the semisimplicity of $\boldsymbol{PA}_I$'s gives that every $\boldsymbol{PA}_I$ is representable.

5. **The amalgamation property.** The constructions of this section follow very closely some work of Daigneault [2] who proved the amalgamation property for locally finite algebras. The greatest difficulty which arises in generalizing Daigneault's work is encountered in some rather complicated constructions which are required to make transfinitely recursive constructions. Some of this has already been done in §3.

Before proceeding, let us give a precise statement of the amalgamation property. If $\boldsymbol{K}$ is a class of similar algebras, then $\boldsymbol{K}$ has the *amalgamation property* provided for any $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \boldsymbol{K}$ and monomorphisms $f_i\colon \mathfrak{A} \to \mathfrak{B}_i$ for $i=1, 2$, there is a $\mathfrak{C} \in \boldsymbol{K}$ and monomorphisms $g_i\colon \mathfrak{B}_i \to \mathfrak{C}$ for $i=1, 2$ such that $g_1 f_1 = g_2 f_2$. $\boldsymbol{K}$ has the *strong amalgamation property* if $\mathfrak{C}$, g_1, g_2 can be taken in such a way that

$$\operatorname{rng}(g_1) \cap \operatorname{rng}(g_2) = \operatorname{rng}(g_1 f_1).$$

In this section we show that for I infinite, the class of all $\boldsymbol{PA}_I$'s and the class of all $\boldsymbol{PEA}_I$'s have the strong amalgamation property. This is in contrast to Comer [1] where it is shown for $2 \leqq |I| < \omega$, the class of $\boldsymbol{PA}_I$'s and the class of $\boldsymbol{PEA}_I$'s do not have the amalgamation property. By the results of §2 it is easy to see that the class of all $\boldsymbol{PA}_I$'s ($\boldsymbol{PEA}_I$'s) has the (strong) amalgamation property for every infinite I if and only if the class of all $\boldsymbol{PA}_I$'s ($\boldsymbol{PEA}_I$'s) of local degree at most $\mathfrak{M}$ has this property for all infinite $\mathfrak{M}$ and all I with $\mathfrak{M} \leqq |I|$. This is done by applying the functorial properties of dilations and compressions. Thus we will assume throughout this section that I is a fixed infinite set and that every algebra referred to has local degree $\mathfrak{M} \leqq |I|$. Then, as indicated, the main results remain true without this restriction.

Suppose $\mathfrak{A}$ is a $\boldsymbol{PA}_I$, $\mathfrak{B} \subseteq \mathfrak{A}$, T is a Y-constant on $\mathfrak{A}$ and $X \subseteq Y$; then we define

$$B^{\mathfrak{A}}(X) = \{T(y/J)p : y \in {}^I X,\ J \subseteq I,\ p \in B,\ y \restriction J \text{ is one to one}\}$$

and

$$\mathfrak{B}^{\mathfrak{A}}(X) = \langle B^{\mathfrak{A}}(X), +, \cdot, -, 0, 1, S(\tau), \exists(J)\rangle.$$

(Recall here our convention that τ runs over all ${}^I I$ and J runs over $S(J)$.) When no confusion is likely, we shall write $B(X)$ and $\mathfrak{B}(X)$ for $B^{\mathfrak{A}}(X)$ and $\mathfrak{B}^{\mathfrak{A}}(X)$.

LEMMA 5.1. *$\mathfrak{B}(X)$ is a subalgebra of $\mathfrak{A}$.*

Proof. First observe that in the definition of $\mathfrak{B}(X)$, the restriction that $y \restriction J$ be one to one may be dropped without changing $B(X)$. For suppose $y \restriction J$ is not one to one; choose $J' \subseteq J$ with $y \restriction J'$ one to one and $y(J) = y(J')$. Take $\sigma \in {}^I I$ so that $\sigma \restriction I \sim J = \delta_{I \sim J}$, $\sigma(J) \subseteq J'$ and $y\sigma \restriction J = y \restriction J$; then $T(y/J)p = T(y/J')S(\sigma)p$ by (C5). Obviously $B(X)$ is closed under $-$ and it is closed under $+$ by an application of 3.5. $B(X)$ is closed under $\exists(K)$ because $\exists(K)T(y/J)p = T(y/J)\exists(K \sim J)p$. To show $B(X)$ is closed under $S(\tau)$ suppose $p \in B$, $y \in {}^I X$, and $J \subseteq I$. We need to show $S(\tau)T(y/J)p \in B(X)$. Let K be a support of p with $|K| < \mathfrak{M}$. Since $T(y/J)p = T(y/J \cap K)p$ we may assume $J \subseteq K$. Choose $\tau' \in {}^I I$ such that $\tau' \restriction K \sim J = \tau \restriction K \sim J$, $\tau' \restriction J$ is one to one, and $\tau'(I \sim J) \cap \tau'(J) = 0$. Take $\sigma \in {}^I I$ with $\sigma\tau' \restriction J = \delta_J$. Then we have

$$\begin{aligned} S(\tau)T(y/J)p &= S(\tau')T(y/J) \\ &= S(\tau')T(y\sigma\tau'/J)p = T(y\sigma/\tau' J)S(\tau')p. \end{aligned}$$

This last is an element of $B(X)$, so the lemma is proven.

LEMMA 5.2. *Let $\mathfrak{B}$, $\mathfrak{C} \subseteq \mathfrak{A}$ where $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ and let f be an isomorphism from $\mathfrak{B}$ onto $\mathfrak{C}$. Suppose Y is a set disjoint from I and $\mathfrak{A}'$ is the algebra with Y-constant T obtained by fixing the variables Y in a minimal $I \cup Y$-dilation of $\mathfrak{A}$. Finally let X, $Z \subseteq Y$ and t be a one to one function from X onto Z. Then f' defined on $\mathfrak{B}(X)$ by $f'T(y/J)p = T(ty/J)fp$ for $p \in B$, $y \in {}^I X$, and $J \subseteq I$ is an isomorphism from $\mathfrak{B}(X)$ onto $\mathfrak{C}(Z)$.*

Proof. By 2.9, $\mathfrak{B}(X)$ is an extension of $\mathfrak{B}$ with X-constant T obtained by fixing the variables X in a minimal $I \cup X$-dilation of $\mathfrak{B}$. Thus by 3.10, f' is a well-defined homomorphism from $\mathfrak{B}(X)$ into $\mathfrak{C}(Z)$. By similar reasoning, there is a homomorphism g from $\mathfrak{C}(Z)$ into $\mathfrak{B}(X)$ such that $gT(y/J)p = T(t^{-1}y/J)f^{-1}p$ for $p \in A$, $y \in {}^I Z$, and $J \subseteq I$. Obviously g is an inverse to f'; thus f' is an isomorphism.

Let us introduce some special notions. A *triple* will mean a triple $(\mathfrak{A}, Y, T)$ where $\mathfrak{A}$ is a $\boldsymbol{PA}_I$ and T is a Y-constant on $\mathfrak{A}$. We will also permit the case where $Y = 0$ (hence $T = 0$). If $(\mathfrak{A}, Y, T)$ and $(\mathfrak{B}, Z, T')$ are two triples, by a homomorphism from $(\mathfrak{A}, Y, T)$ to $(\mathfrak{B}, Z, T')$ we mean a pair (f, t) such that f is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, t is a function from Y to Z, and for all $p \in A$, $y \in {}^I Y$, and $J \subseteq I$, $fT(y/J)p = T'(ty/J)fp$. This will be denoted in symbols by $(f, t)\colon (\mathfrak{A}, Y, T) \to (\mathfrak{B}, Z, T')$. $(\mathfrak{A}, Y, T)$ is called a subtriple of $(\mathfrak{B}, Z, T')$—in symbols, $(\mathfrak{A}, Y, T) \subseteq (\mathfrak{B}, Z, T')$ if (δ_A, δ_Y) is a homomorphism from $(\mathfrak{A}, Y, T)$ into $(\mathfrak{B}, Z, T')$. In this case, $\mathfrak{A} \subseteq \mathfrak{B}$, $Y \subseteq Z$, and T is the restriction to $\mathfrak{A}$ of the Y-reduction of T'. We will also say that $(\mathfrak{B}, Z, T')$ is an extension of $(\mathfrak{A}, Y, T)$. In this case we will often write $(\mathfrak{A}, Y, T') \subseteq (\mathfrak{B}, Z, T')$ rather than distinguish between T' and T; this should cause no confusion. A triple $(\mathfrak{A}, Y, T)$ is rich if every element of $\mathfrak{A}$ has a witness in ${}^I Y$. Then by 4.4, every triple has a rich extension. A homomorphism of triples (f, t) is a monomorphism, epimorphism, or isomorphism if and only if both f and t are one to one, onto, or both respectively.

We need extensions of 3.14 and 3.15. These are contained in the next two lemmas. For this suppose we have the following: η is an ordinal; for each $\alpha < \eta$, $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ is a triple, and suppose that whenever $\alpha < \beta < \eta$, $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha) \subseteq (\mathfrak{A}_\beta, Y_\alpha, T_\beta)$. Further suppose for each $\alpha < \eta$ that $(\mathfrak{B}_\alpha, X_\alpha, T_\alpha)$ and $(\mathfrak{C}_\alpha, Z_\alpha, T_\alpha)$ are subtriples of $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$; (f_α, t_α) is an isomorphism from $(\mathfrak{B}_\alpha, X_\alpha, T_\alpha)$ to $(\mathfrak{C}_\alpha, Z_\alpha, T_\alpha)$ and for $\alpha < \beta < \eta$, $(\mathfrak{B}_\alpha, X_\alpha, T_\alpha) \subseteq (\mathfrak{B}_\beta, X_\beta, T_\beta)$, $(\mathfrak{C}_\alpha, Z_\alpha, T_\alpha) \subseteq (\mathfrak{C}_\beta, Z_\beta, T_\beta)$, $f_\alpha \subseteq f_\beta$, and $t_\alpha \subseteq t_\beta$. Take $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, Y, X, Z, f and t to be the union over all $\alpha < \eta$ of $\mathfrak{A}_\alpha$, $\mathfrak{B}_\alpha$, $\mathfrak{C}_\alpha$, Y_α, X_α, Z_α, f_α, and t_α respectively.

LEMMA 5.3. *If η is a regular cardinal with $\mathfrak{M} \leqq \eta$, then there is a Y-constant T on $\mathfrak{A}$ such that for each $\alpha < \eta$, $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha) \subseteq (\mathfrak{A}, Y, T)$, $\mathfrak{B}(X) = \mathfrak{B}$, $\mathfrak{C}(Z) = \mathfrak{C}$, and (f, t) is an isomorphism from $(\mathfrak{B}, X, T)$ onto $(\mathfrak{C}, Z, T)$.*

Proof. Follows immediately from 3.14.

LEMMA 5.4. *Assume we have the situation immediately preceding 5.3. There is an algebra $\mathfrak{A}^*$ with Y-constant T such that for each $\alpha < \eta$, $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha) \subseteq (\mathfrak{A}^*, Y, T)$*

and there is an extension f^* of f such that (f^*, t) is an isomorphism of $(\mathfrak{B}(X), X, T)$ onto $(\mathfrak{C}(Z), Z, T)$. In this case, (f^*, t) extends (f_α, t_α) for all $\alpha<\eta$.

Proof. We proceed as in the proof of 3.15 and use the notation there. Let ν be the natural projection from $\mathfrak{A}'$ onto $\mathfrak{A}^*$. Let $\bar{\mathfrak{A}}$, $\bar{\mathfrak{B}}$, $\bar{\mathfrak{C}}$ be the subalgebras of $\mathfrak{A}^*$ which are the images under ν of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. Then as by the proof of 3.15, $\nu\restriction A$, $\nu\restriction B$, $\nu\restriction C$ are isomorphisms from $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ onto $\bar{\mathfrak{A}}$, $\bar{\mathfrak{B}}$, $\bar{\mathfrak{C}}$. We write $\mathfrak{B}(X)$, $\mathfrak{C}(Z)$ for $\mathfrak{B}^{\mathfrak{A}'}(X)$, $\mathfrak{C}^{\mathfrak{A}'}(X)$ and $\bar{\mathfrak{B}}(X)$, $\bar{\mathfrak{C}}(Z)$ for $\mathfrak{B}^{\mathfrak{A}^*}(X)$, $\mathfrak{C}^{\mathfrak{A}^*}(Z)$. Notice that (ν, δ_Y) maps $(\mathfrak{B}(X), X, T')$ and $(\mathfrak{C}(Z), Z, T')$ onto $(\bar{\mathfrak{B}}(X), X, T)$ and $(\bar{\mathfrak{C}}(Z), Z, T)$ respectively.

By 5.2, there is an isomorphism f' from $\mathfrak{B}(X)$ onto $\mathfrak{C}(Z)$ such that (f', t) is an isomorphism from $(\mathfrak{B}(X), X, T)$ onto $(\mathfrak{C}(Z), Z, T)$. We define f^* by $f^*\nu p=\nu f'p$ for $p\in B(X)$. To see f^* is well defined, suppose $q\in B(X)$ and $\nu q=0$; we must show that $\nu f'q=0$. But this is equivalent to showing that $f'q\in M$ under the assumption that $q\in M$. Now $q=T'(x/J)p$ for some $p\in B$, $x\in {}^IX$, $J\subseteq I$, and $x\restriction J$ one to one. Further, by the definition of M and the fact that the generators of M are closed under $+$, there is $\alpha<\eta$, $r\in A_\alpha$, $y\in {}^IY_\alpha$, and $K\subseteq I$ such that $q\leqq T(y/K)r$ and $T_\alpha(y/K)r=0$. We may also suppose $p\in A_\alpha$ since there is β with $p\in A_\beta$ and we can replace α by $\max\{\alpha, \beta\}$. Now let $J_1=\{i\in J: x_i\in Y_\alpha\}$ and $J_2=J\sim J_1$. Let σ be a transformation of $I\cup Y$ with $\sigma x\restriction J_1=\delta_{J_1}$ and σ the identity outside of $x(J_1)$. In $\mathfrak{A}^+$, $T'(y/K)r$ is independent of $x(J_1)$, so $S^+(\sigma)T'(y/K)r=T'(y/K)r$. Also $S^+(\sigma)q=S^+(\sigma)T'(x/J)p=T'(x/J_2)p$. Thus we obtain $T'(x/J_2)p\leqq T'(y/K)r$. Now take $x\in {}^IY_\alpha$ such that $\bar{x}\restriction J_2=x\restriction J_2$; then $T'(\bar{x}/J_2)p=T'(x/J)p\leqq T'(y/K)r$, so $T'(\bar{x}/J_2)\in M$. From this it follows that $T(\bar{x}/J_2)\nu p=\nu T'(\bar{x}/J_2)p=0$, and thus $T_\alpha(\bar{x}/J_2)p=0$. Hence $f_\alpha T_\alpha(\bar{x}/J_2)p=T_\alpha(t\bar{x}/J_2)f_\alpha P=T^*(t\bar{x}/J_2)f_\alpha p=\nu T'(t\bar{x}/J_2)fp=0$. Thus we have $T'(t\bar{x}/J_2)fp\in M$. But then $f'q=T'(tx/J)fp\leqq\exists(J_1)T'(tx/J_2)fp$ $=\exists(J_1)T'(t\bar{x}/J_2)fp\in M$. Thus f^* is well defined. Next, f^* is an isomorphism, because similar considerations with (f'^{-1}, t^{-1}) in place of (f', t) yield an inverse to f^*. Finally, it is clear that for $\alpha<\eta$, $p\in A_\alpha$, that $\nu f_\alpha p=f^*\nu p$. The lemma follows after identifying A with $\bar{A}$ via ν.

LEMMA 5.5. *Suppose $(\mathfrak{A}, X, T)\subseteq(\mathfrak{B}, Y, T)$ and (f, t) is a monomorphism from $(\mathfrak{A}, X, T)$ into $(\mathfrak{B}, Y, T)$, then there is an extension $(\mathfrak{C}, Z, T^*)$ of $(\mathfrak{B}, Y, T)$ and an extension (f', t') of (f, t) which is a monomorphism from $(\mathfrak{A}(Y), Y, T)$ into $(\mathfrak{C}, Z, T^*)$. If $\mathfrak{B}$ is simple, $\mathfrak{C}$ can be taken to be simple.*

Proof. Let r: $Y\sim X\to W$ be one to one and onto and assume $W\cap(I\cup Y)=0$. Take $\mathfrak{D}$ to be the algebra obtained by fixing the variables W in a minimal $I\cup W$-dilation $\mathfrak{B}^+$ of $\mathfrak{B}$, and let T' be the $Y\cup W$-constant of $\mathfrak{D}$ obtained by applying 3.11 to the extension of the Y-constant T to $\mathfrak{D}$ and the W-constant on $\mathfrak{D}$ obtained by fixing variables. Take $Z=Y\cup W$; then $(\mathfrak{B}, Y, T)\subseteq(\mathfrak{D}, Z, T')$. Now let M be the ideal of $\mathfrak{D}$ generated by

$$\{T'(rx/J)fp: p\in A,\ x\in {}^I(Y\sim X),\ J\subseteq I, \text{ and } T(x/J)p=0\}.$$

Then by methods similar to those used in 3.15, $\mathrm{B} \cap M = \{0\}$. Take $\mathfrak{C} = \mathfrak{D}/M$ and let T^* be the Z-constant of $\mathfrak{C}$ induced by T. Then we can consider $(\mathfrak{B}, Y, T)$ to be contained in $(\mathfrak{C}, Z, T^*)$. Now for $q \in A(Y)$, there is $p \in A$, $y \in {}^I(Y \sim X)$, $J \subseteq I$ with $q = T(y/J)p$; define $f'q = T^*(ry/J)fp$. Further take $t' = t \cup r$; now by methods similar to those used in 3.15, f' is a well-defined monomorphism and $f'T(y/J)p = T(t'y/J)f'p$ for $p \in A(Y)$, $y \in {}^I Y$, and $J \subseteq I$. Thus (f', t') is the required monomorphism. Finally, if $\mathfrak{B}$ is simple, $\mathfrak{C}$ is obtained simple by extending M to a maximal ideal of $\mathfrak{D}$.

LEMMA 5.6. *Let* $(\mathfrak{B}, X, T)$, $(\mathfrak{C}, Z, T) \subseteq (\mathfrak{A}, Y, T)$ *and* (f, t) *be an isomorphism from* $(\mathfrak{B}, X, T)$ *onto* $(\mathfrak{C}, Z, T)$. *Then there is a rich extension* $(\mathfrak{A}^*, Y^*, T^*)$ *of* $(\mathfrak{A}, Y, T)$ *and an extension of* (f, t) *to an isomorphism* (f^*, t^*) *from* $(\mathfrak{B}(Y^*), Y^*, T^*)$ *onto* $(\mathfrak{C}(Y^*), Y^*, T^*)$. *If* $\mathfrak{A}$ *is simple,* $\mathfrak{A}^*$ *can be taken simple.*

Proof. Let $\mathfrak{N}$ be the first regular cardinal greater than or equal to $\mathfrak{M}$. By transfinite recursion we define for each $\alpha < \mathfrak{N}$ a triple $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ with two subtriples $(\mathfrak{B}_\alpha, X_\alpha, T_\alpha)$ and $(\mathfrak{C}_\alpha, Z_\alpha, T_\alpha)$ and an isomorphism $(f_\alpha, t_\alpha): (\mathfrak{B}_\alpha, X_\alpha, T_\alpha) \to (\mathfrak{C}_\alpha, Z_\alpha, T_\alpha)$ as follows. $(\mathfrak{A}_0, Y_0, T_0) = (\mathfrak{A}, Y, T)$, $(\mathfrak{B}_0, X_0, T_0) = (\mathfrak{B}, X, T)$, $(\mathfrak{C}_0, Z_0, T_0) = (\mathfrak{C}, Z, T)$, and $(f_0, t_0) = (f, t)$. Having made the definition for α, if α is even, we take $X_{\alpha+1} = Y_\alpha$ and apply 5.5 to obtain $(\mathfrak{A}_{\alpha+1}, Y'_{\alpha+1}, T_{\alpha+1})$ and a monomorphism $(f_{\alpha+1}, t_{\alpha+1})$ from $(\mathfrak{B}_\alpha(Y_\alpha), Y_\alpha, T_\alpha)$ into $(\mathfrak{A}'_{\alpha+1}, Y_{\alpha+1}, T_{\alpha+1})$ which extends (f_α, t_α). We then let $(\mathfrak{A}_{\alpha+1}, Y_{\alpha+1}, T_{\alpha+1})$ be a rich extension of $(\mathfrak{A}'_{\alpha+1}, Y'_{\alpha+1}, T_{\alpha+1})$, $\mathfrak{B}_{\alpha+1} = \mathfrak{B}_\alpha(X_{\alpha+1})$, $\mathfrak{C}_{\alpha+1} = f_{\alpha+1}(\mathfrak{B}_{\alpha+1})$, and $Z_{\alpha+1} = t_{\alpha+1}(X_{\alpha+1})$. If α is odd, we go through the same procedure with the roles of $\mathfrak{B}_\alpha$ and $\mathfrak{C}_\alpha$ reversed and $(f_\alpha^{-1}, t_\alpha^{-1})$ in place of (f_α, t_α). For α a limit ordinal we apply 5.4 to obtain $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ an extension of $(\mathfrak{A}_\beta, Y_\beta, T_\beta)$ for all $\beta < \alpha$ with $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ and to obtain (f_α, t_α) an isomorphism from $(\mathfrak{B}_\alpha, X_\alpha, T_\alpha)$ onto $(\mathfrak{C}_\alpha, Z_\alpha, T_\alpha)$ where $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$, $\mathfrak{B}_\alpha = (\bigcup_{\beta<\alpha} \mathfrak{B}_\beta(X_\alpha))$ and similarly for $\mathfrak{C}_\alpha$ and Z_α. Then (f_α, t_α) is taken to extend (f_β, t_β) for each $\beta < \alpha$. We take $\mathfrak{A}^* = \bigcup_{\alpha<\mathfrak{N}} \mathfrak{A}_\alpha$. $Y^* = \bigcup_{\alpha<\mathfrak{N}} Y_\alpha$. Notice $Y^* = \bigcup_{\alpha<\mathfrak{N}} X_\alpha = \bigcup_{\alpha<\mathfrak{N}} Z_\alpha$, thus by 4.14 there is a Y^*-constant T^* on $\mathfrak{A}^*$ such that $(\mathfrak{A}^*, Y^*, T^*)$ is an extension of $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ for each $\alpha < \mathfrak{N}$ and an extension (f^*, t^*) of (f_α, t_α) to an isomorphism from $(\mathfrak{B}^*, Y^*, T^*)$ onto $(\mathfrak{C}^*, Y^*, T^*)$ where $\mathfrak{B}^* = \bigcup_{\alpha<\mathfrak{N}} \mathfrak{B}_\alpha$ and $\mathfrak{C}^* = \bigcup_{\alpha<\mathfrak{N}} \mathfrak{C}_\alpha$. We will be done if we show $\mathfrak{B}^* = \mathfrak{B}(Y^*)$ and $\mathfrak{C}^* = \mathfrak{C}(Y^*)$. These are similar and one inclusion is obvious, so we will only show that $\mathfrak{B}^* \subseteq \mathfrak{B}(Y^*)$. Suppose $p \in B^*$; then for some $\alpha < \mathfrak{N}$, $p \in B_\alpha$. We show by induction on α that $p \in B(Y^*)$. This is obvious if $\alpha = 0$. Now suppose we have shown it for all $\beta < \alpha$. Then we have by the definition of $\mathfrak{A}_\alpha$ that $p = T_\beta(y/J)q$ for some $q \in B_\beta$ for some $\beta < \alpha$ and some $y \in {}^I Y_\alpha$, $J \subseteq I$. Now by the induction hypothesis, $q = T^*(z/K)r$ for some $r \in B$, $z \in {}^I Y^*$, and $K \subseteq I$. Take $x \in {}^I Y^*$ so that $x \restriction K = z \restriction K$ and $x \restriction J \sim K = y \restriction J \sim K$. Then $p = T_\beta(y/J)T^*(z/K)r = T^*(y/J \sim K)T^*(x \sim K)r = T^*(x/J \cup K)r$. Thus $p \in B(Y^*)$. If $\mathfrak{A}$ is simple, we obtain $\mathfrak{A}^*$ simple by taking $\mathfrak{A}_\alpha$ simple for each $\alpha < \mathfrak{N}$. $(\mathfrak{A}^*, Y^*, T^*)$ is rich since for each α, $(\mathfrak{A}_{\alpha+1}, Y_{\alpha+1}, T_{\alpha+1})$ is a rich extension of $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$. This completes the proof of 5.6.

LEMMA 5.7. *Let* $(\mathfrak{B}, X, T)$ *and* $(\mathfrak{C}, Z, T)$ *be subtriples of* $(\mathfrak{A}, Y, T)$ *where* $\mathfrak{A}$ *is simple. Suppose* (f, t) *is an isomorphism from* $(\mathfrak{B}, X, T)$ *onto* $(\mathfrak{C}, Z, T)$. *Then there is an extension* $(\mathfrak{A}^*, Y^*, T^*)$ *of* $(\mathfrak{A}, Y, T)$ *such that* (f, t) *extends to an automorphism* (f^*, t^*) *of* $(\mathfrak{A}^*, Y^*, T^*)$.

Proof. By 5.6 let $(\bar{\mathfrak{A}}, \bar{Y}^*, T)$ be a rich extension of $(\bar{\mathfrak{A}}, Y, T)$ with $\bar{\mathfrak{A}}$ simple and such that $(\bar{f}, t^*)$ is an extension of (f, t) to an isomorphism from $(\bar{\mathfrak{B}}, Y^*, \bar{T})$ onto $(\bar{\mathfrak{C}}, Y^*, \bar{T})$ where $=\mathfrak{C}(\bar{\mathfrak{C}}Y^*)$ and $\bar{\mathfrak{B}}=\mathfrak{B}(Y^*)$. Let $\mathfrak{A}^*$ be the $\boldsymbol{PSA}_I$ of all subsets of ${}^I Y$ which have a support of power less than $\mathfrak{M}$ and let T^* be the functional Y^*-constant on $\mathfrak{A}^*$ (cf. Example 1 of §3). For $p \in \bar{A}$ define

$$gp = \{y \in {}^I Y : \bar{T}(y/I)p = 1\}.$$

Then by 4.6, (g, δ_Y) is a monomorphism from $(\bar{\mathfrak{A}}, Y^*, \bar{T})$ into (A^*, Y^*, T^*). Now let f^* be the automorphism of $\mathfrak{A}^*$ given in 1.11 which is induced by t^*, i.e., for $M \in A^*$, $f^*M=\{t^*x : x \in M\}$. Then it is easily seen that (f^*, t^*) is an automorphism of (A^*, Y^*, T^*). The only thing left to show is that for $p \in B$, $gfp=f^*gp$. For this we have

$$\begin{aligned} y \in f^*gp \quad &\text{iff} \quad t^{*-1}y \in gp \\ &\text{iff} \quad \bar{T}(t^{*-1}y/I)p = 1 \\ &\text{iff} \quad \bar{T}(y/I)fp = 1 \\ &\text{iff} \quad y \in gfp. \end{aligned}$$

The lemma then follows by identifying $\mathfrak{A}$ with its image under g.

LEMMA 5.8. *Suppose for each* $\alpha \in M$ *where* M *is any set* $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ *is a triple, then there is a triple* $(\mathfrak{A}, Y, T)$ *such that for each* $\alpha \in M$, *there is a monomorphism* (f_α, t_α) *from* $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ *into* $(\mathfrak{A}, Y, T)$.

Proof. By 4.4 we may assume each of the triples $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ is rich. Let $Y=\bigcup_{\alpha\in M} Y_\alpha$ and for each α let $u_\alpha\colon Y \to Y_\alpha$ such that $u_\alpha \restriction Y_\alpha = \delta_{Y_\alpha}$. Then T'_α defined by $T'_\alpha(y/J)p=T_\alpha(u_\alpha y/J)p$ for $p \in A_\alpha$, $y \in {}^I Y$, $J \subseteq I$ is a Y-constant on $\mathfrak{A}_\alpha$ and $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha) \subseteq (\mathfrak{A}_\alpha, Y, T'_\alpha)$. Let $\mathfrak{A}$ be the $\boldsymbol{PSA}_I$ with base Y and let L be the set of all functions on M such that for $l \in L$ and $\alpha \in M$, l_α is a Boolean ultrafilter on $\mathfrak{A}_\alpha$. For each $\alpha \in M$ define $f_\alpha\colon \mathfrak{A}_\alpha \to {}^L\mathfrak{A}$ by $(f_\alpha p)_l=\{y \in {}^I Y : T'(y/I)p \in l_\alpha\}$. Then, by 4.6, f_α is a monomorphism from $\mathfrak{A}_\alpha$ into ${}^L\mathfrak{A}$. Next observe that if T' is the functional Y-constant on $\mathfrak{A}$ (cf. §3, Example 1), then T defined by $(T(y/J)p)_l=T'(y/J)p_l$ for $p \in {}^L A$, $y \in {}^I Y$, and $J \subseteq I$ is a Y-constant on ${}^L\mathfrak{A}$. It is then easy to see that for each $(f_\alpha, \delta_{Y_\alpha})$ is a monomorphism of $(\mathfrak{A}_\alpha, Y_\alpha, T_\alpha)$ into $({}^L\mathfrak{A}, Y, T)$.

We are now ready to prove our first main theorem.

THEOREM 5.9. *The class of all* $\boldsymbol{PA}_I$*'s has the amalgamation property. In fact, the class of all triples has the amalgamation property.*

Proof. The first statement follows from the second. Suppose for $i=1, 2$ that (f_i, t_i) is a monomorphism from $(\mathfrak{A}, X, T)$ into $(\mathfrak{B}_i, Y_i, T_i)$. First we assume that

the $\mathfrak{B}_i$ are simple. By 5.8 let (g_i, s_i) be monomorphisms from $(\mathfrak{B}_i, Y_i, T_i)$ into a triple $(\mathfrak{C}, Z, T')$ where $\mathfrak{C}$ is simple. By 5.7 let $(\mathfrak{D}, Z^*, T^*)$ be an extension of $(\mathfrak{C}, Z, T')$ which admits an automorphism (h, r) extending $(g_1 f_1 f_2^{-1} g_2^{-1}, s_1 t_1 t_2^{-1} s_2^{-1})$. Take $g_2' = hg_2$ and $s_2' = rs_2$. Then (g_1, s_1) and (g_2', g_2') are monomorphisms which amalgamate (f_1, t_1) and (f_2, t_2) i.e. $(g_1 f_1, s_1 t_1) = (g_2' f_2, s_2' t_2)$.

Now to prove the general case, let L consist of all pairs (M, N) such that M is a maximal ideal on $\mathfrak{B}_1$, N is a maximal ideal on $\mathfrak{B}_2$, and $f_1^{-1}(M) = f_2^{-1}(N)$. For each $l \in L$, let f_1^l be the monomorphism induced by f_1 from $\mathfrak{A}/f_1^{-1}(M)$ into $\mathfrak{B}_1/M$ where $l = (M, N)$. Similarly define f_2^l from $\mathfrak{A}/f_2^{-1}(N)$ into $\mathfrak{B}_2/N$. Let $(\mathfrak{C}^l, Y^l, T^l)$ be a triple and, for $i = 1, 2$, (g_i^l, s_i^l) a monomorphism from $(\mathfrak{B}_i/M, Y_i, T_i)$ into $(\mathfrak{C}^l, Y^l, T^l)$ such that $(g_1^l f_1^l, s_1^l t_1^l) = (g_2^l f_2^l, s_2^l t_2^l)$. We may assume $Y^l = Y^{l'}$ and $s_i^l = s_i^{l'}$ for all $l, l' \in L$, say $Y^l = Z$, $s_i^l = s_i$. In more detail: Let $Z = \mathsf{P}_{l \in L} Y^l$ and s_i the function from Y_i into Z defined by $s_i(y)_l = s_i^l(y)$ for $y \in Y_i$. Now let pr_l be the natural projection of Z onto Y^l. For $z \in {}^I Z$, $J \subseteq I$, and $p \in C^l$, define $\overline{T}^l(z/J)p = T^l(\mathrm{pr}_l\, z/J)p$. Then it is easy to see that $\overline{T}^l$ is a Z-constant on $\mathfrak{C}^l$. Thus we do make the noted assumptions. Define for $i = 1, 2$, g_i from $\mathfrak{B}_i$ into $\mathfrak{C} = \mathsf{P}_{l \in L} \mathfrak{C}^l$ by $(g_1 p)_l = g_1^l(p/M)$ and $(g_2 q)_l = g_2^l(q/N)$ where $l = (M, N)$, $p \in B_1$, $q \in B_2$. Then (g_i, s_i) is a monomorphism from $(\mathfrak{B}_i, Y_i, T_i)$ into $(\mathfrak{C}, Z, T^*)$ where $(T^*(y/J)p)_l = (T^l(y/J)p)_l$ whenever $l \in L$, $p \in C$, $y \in {}^I Z$, and $J \subseteq I$. Further it is clear that $(g_1 f_1, s_1 t_1) = (g_2 f_2, s_2 t_2)$. This completes the proof.

It would have been slightly easier to prove 5.9 without considering the amalgamation of triples. However, we need the stronger form to prove the next theorem.

THEOREM 5.10. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are $\boldsymbol{PA}_I$'s with $\mathfrak{A} \subseteq \mathfrak{B}$; let $p \in B \sim A$, then there is a $\boldsymbol{PA}_I$ $\mathfrak{C}$, and there are homomorphisms f_1 and f_2 from $\mathfrak{B}$ into $\mathfrak{C}$ such that $f_1 \restriction A = f_2 \restriction A$ and $f_1 p \neq f_2 p$.*

Proof. First assume the local degree of $\mathfrak{B}$ is $\mathfrak{M} \leqq |I|$. Let Y be a set disjoint from I with y a one to one function from I onto Y. Let $(\mathfrak{B}', Y, T)$ be the triple obtained by fixing the variables Y in a minimal $I \cup Y$-dilation of B and let $(\mathfrak{A}', Y, T)$ be the subtriple of $(\mathfrak{B}', Y, T)$ with $\mathfrak{A}' = \mathfrak{A}(Y)$. Now let M_1 be the ideal of B' generated by $T(y/I)p$ and M_2 be the ideal of B' generated by $T(y/I)(-p)$. Notice that $\exists (I) T(y/I)p = T(y/I)p$ and similarly for $T(y/I)(-p)$. Next we claim $T(y/I)p \notin A'$ and $T(y/I)(-p) \notin A'$. Since these are similar we consider only the case $T(y/I)p$. Suppose $T(y/I)p \in A'$; then $T(y/I)p = T(x/J)q$ for some $q \in A$, $x \in {}^I Y$, and $J \subseteq I$ with $|J| < \mathfrak{M}$ and $x \restriction J$ one to one. Let σ be a transformation of $I \cup Y$ such that $\sigma \restriction I = \delta_I$, $\sigma x \restriction J = \delta_J$, and σ maps Y one to one onto I. Then we have $q = S(\delta_I)q = S^+(\sigma)T(x/J)q = S^+(\sigma)T(y/I)p = S(\sigma y)p$. From this we obtain $p = S((\sigma y)^{-1})q \in A$ which is a contradiction. Now by a theorem of Daigneault (4.1 of [2]) there are maximal ideals N_1, N_2 of $\mathfrak{B}'$ such that $N_1 \cap A' = N_2 \cap A'$ and $M_i \subseteq N_i$ for $i = 1, 2$. Let $N = N_1 \cap A'$. Thus we may take ν to the natural monomorphism from $\mathfrak{A}'/N$ both $\mathfrak{B}/N_1$ and $\mathfrak{B}/N_2$ by $\nu(p/N) = p/N_i$ for $i = 1, 2$. This gives monomorphisms (ν, δ_Y) from $(\mathfrak{A}'/N, Y, T)$ into $(\mathfrak{B}/N_i, Y, T)$ for $i = 1, 2$. By 6.9 let $(\mathfrak{C}, Z, T^*)$ be a

triple such that for $i=1, 2$ there are monomorphisms (g_i, t_i) from $(\mathfrak{B}/N_i, Y, T)$ in $(\mathfrak{C}, Z, T^*)$ with the property that $(g_1\nu, t_1)=(g_2\nu, t_2)$. Notice then $t_1=t_2$. For $i=1, 2$ let f_i be the composition of g_i with the natural homomorphism from $\mathfrak{B}$ onto $\mathfrak{B}/N_i$. Then clearly $f_1\restriction A=f_2\restriction A$. Finally we have $f_1p\neq f_2p$ for $T^*(t_1y/I)f_1p=f_1T(y/I)p=f_10=0$ while $T^*(t_1y/I)f_2p=T^*(t_2y/I)f_2p=f_2T(y/I)p=f_21=1$.

Now we claim the general case follows from the case with restricted local degree condition. For this, let I^+ be a superset of I with $|I|<|I^+|$, $\mathfrak{B}^+$ be a minimal I^+-dilation of $\mathfrak{B}$, and $\mathfrak{A}^+\subseteq\mathfrak{B}^+$ the minimal I^+-dilation of $\mathfrak{A}$ contained in $\mathfrak{B}$. First notice that $p\notin A^+$; for if $p\in A^+$, then by 2.3(i), $p=S^+(\sigma)q$ for some $q\in A$ and $\sigma\in{}^{I^+}I^+$. Take $\tau\in{}^{I^+}I^+$ so that $\tau\restriction I=\delta_I$ and $\tau\sigma$ maps I into I. Now since $p\in B$, I supports p; thus $p=S^+(\tau)p=S^+(\tau\sigma)q=S(\tau\sigma\restriction I)q$. Then $p\in A$, contradicting the assumption that $p\in B\sim A$. Now the theorem follows easily from the first part of the proof and 2.8.

COROLLARY 5.11. *The class of all* $\boldsymbol{PA}_I$*'s has the strong amalgamation property.*

Proof. Suppose $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ are $\boldsymbol{PA}_I$'s and, for $i=1, 2$, f_i is a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}_i$. By 5.9 let g_i be a monomorphism from $\mathfrak{B}_i$ into a $\boldsymbol{PA}_I$ C such that $g_1f_1=g_2f_2$. Let $L=\operatorname{rng}(g_1)\cap\operatorname{rng}(g_2)\sim\operatorname{rng}(g_1f_1)$, and let $\mathfrak{A}'=g_1f_1(\mathfrak{A})$. For each $p\in L$ let h_1^p and h_2^p be homomorphisms of $\mathfrak{C}$ into a $\boldsymbol{PA}_I$ $\mathfrak{D}_p$ such that $h_1^p\restriction A'=h_2^p\restriction A'$ and $h_1^pp\neq h_2^pp$. For $i=1, 2$ define g_i' a homomorphism from $\mathfrak{B}_i$ into $\mathsf{P}_{p\in L}\,\mathfrak{D}_p$ by $(g_i'q)p=h_i^pg_iq$. Then we have $g_1'f_1=g_2'f_2$; the only problem is that g_1' and g_2' may not be monomorphisms. To overcome this difficulty, let g_i'' be a homomorphism from $\mathfrak{B}_i$ into $\mathfrak{C}\times\mathsf{P}_{p\in L}\,\mathfrak{D}_p$ defined by $g_i''q=(g_iq, g_i'q)$ for $i=1, 2$. Then g_1'', g_2'' are monomorphisms, $g_1''f_1=g_2''f_2$, and $\operatorname{rng}(g_1'')\cap\operatorname{rng}(g_2'')$ coincides with $\operatorname{rng}(g_1''f_1)$.

It follows immediately from 5.9 that the class of simple $\boldsymbol{PA}_I$'s (and hence the class of $\boldsymbol{PSA}_I$'s) has the amalgamation property. The corresponding question for the strong amalgamation property remains open, though we conjecture that the answer is negative there.

Turning now to the case of equality algebras we have

THEOREM 5.12. *The class of* $\boldsymbol{PEA}_I$*'s has the strong amalgamation property.*

The proof of 5.12 follows exactly along the lines of the same result for locally finite algebras which can be found in §3 of [2]. The only things to bear in mind are that the theorem must be proven first for algebras of local degree $\mathfrak{M}\leqq|I|$ and then generalized via the dilation and compression theorems of §3, and that a stronger prenex normal form theorem than the one used by Daigneault is required. This last is given now.

THEOREM 5.13 (PRENEX NORMAL FORM THEOREM). *Suppose* $\mathfrak{A}$ *is a* $\boldsymbol{PA}_I$ *(with local degree* $\mathfrak{M}\leqq|I|$*) and* M *is a subset of* A *closed under the Boolean operations and under* $S(\tau)$ *for each* $\tau\in{}^II$. *Then every element of the subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *generated by* M *can be written in the form* $Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p$ *where* $n\in\omega$, $p\in M$, *and for each* $i<n$, Q_i *is either* $\exists$ *or* $\forall$ *and* $J_i\subseteq I$.

Proof. Let C be the set of all elements which can be written in that form. Obviously $M \subseteq C \subseteq B$ so we need to show that C is closed under the operations of A. It is clear that C is closed under $\exists(J)$ and $-$. To see C is closed under $S(\tau)$, suppose $\tau \in {}^I I$ and $Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p \in C$ with $p \in M$. Let K be a support of p with $|K| < \mathfrak{M}$. We may assume $J_0, \ldots, J_{n-1} \subseteq K$ since $Q_0(J_0 \cap K)\cdots Q_{n-1}(J_{n-1} \cap K)p = Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p$. Let $J = J_0 \cup \cdots \cup J_{n-1}$ and take $\sigma \in {}^I I$ such that $\sigma \restriction K \sim J = \tau \restriction K \sim J$, $\sigma \restriction J$ is one to one, and $\sigma(J) \cap \sigma(I \sim J) = 0$. Then applying (P6) (and an obvious induction) we have

$$\begin{aligned} S(\tau)Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p &= S(\sigma)Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p \\ &= Q_0(\sigma J_0)\cdots Q_{n-1}(\sigma J_{n-1})S(\sigma)p. \end{aligned}$$

Hence C is closed under $S(\tau)$.

To see C is closed under $+$, suppose $p, q \in M$, $J_0, \ldots, J_{n-1}, J_n, \ldots, J_{m-1} \subseteq I$ and each of $Q_0, \ldots, Q_{m-1}$ is either $\exists$ or $\forall$. Let K be a support of both p and q with $|K| < \mathfrak{M}$ and assume as above that $J_0 \cup \cdots \cup J_{m-1} \subseteq K$. Let σ and τ be permutations of I such that

$$\begin{aligned} \sigma \restriction K \sim (J_0 \cup \cdots \cup J_{n-1}) &= \delta_{K \sim (J_0 \cup \cdots \cup J_{n-1})}, \\ \tau \restriction K \sim (J_n \cup \cdots \cup J_{m-1}) &= \delta_{k \sim (J_n \cup \cdots \cup J_{m-1})}, \end{aligned}$$

and such that K, $\sigma(J_0 \cup \cdots \cup J_{n-1})$, and $\tau(J_n \cup \cdots \cup J_{m-1})$ are pairwise disjoint. Then an easy calculation gives

$$\begin{aligned} &Q_0(J_0)\cdots Q_{n-1}(J_{n-1})p + Q_n(J_n)\cdots Q_{m-1}(J_{m-1})p \\ &\qquad = Q_0(\sigma J_0)\cdots Q_{n-1}(\sigma J_{n-1})Q_n(\tau J_n)\cdots Q_{m-1}(\tau J_{m-1})(S(\sigma)p + S(\tau)q). \end{aligned}$$

Thus C is closed under the operations of $\mathfrak{A}$ so $\mathfrak{C} = \mathfrak{B}$.

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